MMT-008

ASSIGNMENT BOOKLET

(Valid from 1st July, 2022 to 30th June, 2023)

M.Sc. (Mathematics with Applications in Computer Science)

PROBABILITY AND STATISTICS (MMT-008)

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068

(2022-23)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective courses that we sent you after your enrolment. A weightage of 20 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO. ……………………………

STUDY CENTRE: ……………………………. DATE: ……………………………………….

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto June, 2023. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by June, 2023, then you need to get the assignment for the session 2023-24 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment (MMT – 008)

Course Code: MMT-008
Assignment Code: MMT-008/TMA/2022-23
Maximum Marks: 100

1. State whether the following statements are True or False. Justify your answer with a short proof or a counter example: (10)

   a) If P is a transition matrix of a Markov Chain, then all the rows of $\lim_{n\to\infty} P^n$ are identical.

   b) In a variance-covariance matrix all elements are always positive.

   c) If X_1, X_2, X_3 are iid from $N_2(\mu, \Sigma)$, then $\frac{X_1+X_2+X_3}{3}$ follows $N_2\left(\mu, \frac{1}{3}\Sigma\right)$.

   d) The partial correlation coefficients and multiple correlation coefficients lie between -1 and 1.

   e) For a renewal function M_t, $\lim_{t\to 0} \frac{M_t}{t} = \frac{1}{\mu}$.

2. a) Let (X, Y) have the joint p.d.f. given by: (5)

$$f(x,y) = \begin{cases} 1, & \text{if } |y| < x; 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

   i) Find the marginal p.d.f.'s of X and Y.
   ii) Test the independence of X and Y.
   iii) Find the conditional distribution of X given $Y = y$.
   iv) Compute $E(X \mid Y = y)$ and $E(Y \mid X = x)$.

   b) Let the joint probability density function of two discrete random X and Y be given as: (5)

		X			
		2	3	4	5
Y	0	0	0.03	0	0
	1	0.34	0.30	0.16	0
	2	0	0	0.03	0.14

   i) Find the marginal distribution of X and Y.

ii) Find the conditional distribution of X given Y=1.

iii) Test the independence of variable s X and Y.

iv) Find $V\left[\frac{Y}{X}=x\right]$.

3. a) Let $X \sim N_3(\mu,\Sigma)$, where $\mu=[5,\ \ 3,\ \ 4]'$ and (5)

$$\sum=\begin{pmatrix}2 & 1 & 1\\ 1 & 1 & 0.5\\ 1 & 0.5 & 1\end{pmatrix}.$$

Find the distribution of:

$$\begin{pmatrix}2X_1+X_2-X_3\\ X_1+X_2+X_3\end{pmatrix}$$

b) Determine the principal components Y_1, Y_2 and Y_3 for the covariance matrix: (5)

$$\sum=\begin{pmatrix}1 & -2 & 0\\ -2 & 5 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Also calculate the proportion of total population variance for the first principal component.

4. a) Consider a Markov chain with transition probability matrix: (5)

$$P=\begin{array}{c} \\ 1\\ 2\\ 3\\ 1\end{array}\!\!\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2}\end{pmatrix}\end{array}$$

i) Whether the chain is irreducible? If irreducible classify the states of a Markov chain i.e., recurrent, transient, periodic and mean recurrence time.

ii) Find the limiting probability vector.

b) At a certain filling station, customers arrive in a Poisson process with an average time of 12 per hour. The time interval between service follows exponential distribution and as such the mean time taken to service to a unit is 2 minutes. Evaluate: (5)

i) Probability that there is no customer at the counter.

ii) Probability that there are more than two customers at the counter.

iii) Average number of customers in a queue waiting for service.

iv) Expected waiting time of a customer in the system.

v) Probability that a customer wait for 0.11 minutes in a queue.

5. a) A service station has 5 mechanics each of whom can service a scooter in 2 hours on the average. The scooters are registered at a single counter and then sent for servicing to different mechanics. Scooters arrive at a service station at an average rate of 2 scooters per hour. Assuming that the scooter arrivals are Poisson and service times are exponentially distributed, determine: (5)

i) Identify the model.

ii) The probability that the system shall be idle.

iii) The probability that there shall be 3 scooters in the service centre.

iv) The expected number of scooters waiting in a queue.

v) The expected number of scooters in the service centre.

vi) The average waiting time in a queue.

b) A random sample of 12 factories was conducted for the pairs of observations on sales (x_1) and demands (x_2) and the following information was obtained: (5)

$$\sum X = 96, \sum Y = 72, \sum X^2 = 780, \sum Y^2 = 480, \sum XY = 588$$

The expected mean vector and variance covariance matrix for the factories in the population are:

$$\mu = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

and $\sum = \begin{bmatrix} 13 & 9 \\ 9 & 7 \end{bmatrix}$.

Test whether the sample confirms its truthness of mean vector at 5% level of significance, if:

i) Σ is known,
ii) Σ is unknown.

[You may use: $\chi^2_{2,0.05} = 10.60, \chi^2_{3,0.05} = 12.83,\ \chi^2_{4,0.05} = 14.89, F_{2,10,0.05} = 4.10$]

6. a) Let the random vector $X' = (X_1, X_2, X_3)$ has mean vector $[-2,\ 3,\ 4]$ and variance covariance matrix $= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix}$. Fit the equation $Y = b_0 + b_1 X + b_2 X_2$. Also obtain the multiple correlation coefficient between X_3 and $[X_1, X_2]$. (5)

b) Define ultimate extinction in a branching process. Let $p_k = bc^{k-1}, k = 1, 2, ...;$ $0 < b < c < b + c < 1$ and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Then discuss the probability of extinction in different cases for $E(X_1) \geq 1$ or $E(X_1) < 1$. (5)

7. a) If the random vector Z be $N_4(\mu, \Sigma)$, where: (5)

$$\mu = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

and $\sum = \begin{bmatrix} 3 & 3 & 0 & 9 \\ 3 & 2 & -1 & 1 \\ 0 & -1 & 6 & -3 \\ 9 & 1 & -3 & 7 \end{bmatrix}$.

Find $r_{34}, r_{34.21}$.

b) Suppose life times $X_1, X_2, ...$ are i.i.d. uniformly distributed on $(0,3)$ and $C_1 = 2$ and $C_2 = 8$. Find: (5)

i) μ^T

ii) T which minimizes C (T) and which is the better policy in the long-run in terms of cost.

8. a) Consider the Markov chain with three states, $S = \{1,\ 2,\ 3\}$ following the transition matrix (5)

$$p = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

i) Draw the state transition diagram for this chain.

ii) If $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, then find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

iii) Check whether the chain is irreducible and a periodic.

iv) Find the stationary distribution for the chain.

b) If $N_1(t), N_2(t)$ are two independent Poisson process with parameters λ_1 and λ_2 respectively, then show that (5)

$$P(N_1(t) = k[N_1(t) + N_2(t) = n] = {}^n C_k\, p^k q^{n-k}, \text{ where } p = \frac{\lambda_1}{\lambda_1 + \lambda_2}, q = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

9. a) Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the mean vector $\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Suppose $Y = AX + b$, where (5)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } Y \sim N_3.$$

i) Find $P(0 \le X_2 \le 1)$.

ii) Compute $E(Y)$.

iii) Find the covariance matrix of Y.

iv) Find $P(Y_3 \le 4)$.

b) A box contains two coins: a regular coin and one fake two-headed coin. One coin is chosen at random and tossed twice. The following events are defined: (5)

A: first coin toss results in a head.

B: second coin toss results in a head.

C: coin 1 (regular) has been selected.

Find $P(A|C), P(B|C), P(A \cap B)|C), P(A), P(B)$ and $P(A \cap B)$.

10. a) Consider three random variables X_1, X_2, X_3 having the covariance matrix (5)

$$\begin{bmatrix} 1 & 0.12 & 0.08 \\ 0.12 & 1 & 0.06 \\ 0.08 & 0.06 & 1 \end{bmatrix}.$$

Write the factor model, if number of variables and number of factors are 3 and 1 respectively.

b) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval $[2,5]$ years. Further, planned replacements take place every 3 years. (5)

Compute

i) long-terms rate of replacements.
ii) long-terms rate of failures.