MMT-005

ASSIGNMENT BOOKLET

(Valid from 1st January, 2018 to 31st December, 2018)

M.Sc. (Mathematics with Applications in Computer Science)

COMPLEX ANALYSIS

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2018)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE : …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE : ………………………..….. DATE :……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2018. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by December, 2018, then you need to get the assignment for the year 2019 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the Exam Form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment

Course Code: MMT-005
Assignment Code: MMT-005/TMA/2018
Maximum Marks: 100

1. Determine whether each of the following statement is true or false. Justify your answer with a short proof or a counter example.

   i) $\sin \bar{z}$ is not an analytic function of z anywhere.

   ii) The series $\sum_{n\geq 0} 3^{-n} z^n$ converges uniformly for $|z| < r,\ 0 < r < 3$.

   iii) The function $f(z)$ is analytic in a domain D iff both real and imaginary parts of $f(z)$ and $z\,f(z)$ are harmonic in D.

   iv) for any positive integer $n, |I_m z^n| \leq n\,|I_m z|\,|z|^{n-1}$.

   v) If $\{a_n\}_{n\geq 0}$ is a sequence of real numbers such that

   $$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} a_n (z+2)^n$$

   then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is 3.

   vi) If f is an analytic function on the closed disc $|z| \leq R$ for some fixed positive number $R > 0$, then f can never satisfy the inequality $|f^{(n)}(0)| \geq n!n^n$ for all $n \in N$.

   vii) For $z = x + iy$, $\cosh x$ is never zero and $\cosh z$ has infinitely many zeros when $y \neq 0$.

   viii) The transformation $w = 2^{-1}[z + \alpha^2 z^{-1}]$ $(\alpha \in \mathbb{R})$, maps the circle $|z| = r(r \neq \alpha)$ into a circle in the w-plane.

   ix) If $z = a$ is an isolated essential singularity for $f(z)$, then $z = a$ is neither a regular point nor a pole for $g(z) = 1/f(z)$.

   x) The function $f(z) = \dfrac{e^{1/(z-1)}}{e^{z-1} - 1}$ has a simple pole at $z = 2k\pi i$ $\{k \in \mathbb{Z})$ and an essential singularity at $z = 1$. (20)

2. a) If f is analytic in a neighbourhood N of the origin such that $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1,\ z_2 \in N$, then show that $f(z) = az$ for some complex constant a. (4)

   b) If $a \in \mathbb{R}$ then show that $u(x,\ y) = e^{-2axy} \cos a(x^2 - y^2)$ is harmonic in $\mathbb{R}^2$. Find all its harmonic conjugate $v(x,\ y)$ in $\mathbb{R}^2$. Write $f = u + iv$ as a function of z with $f(0) = 1$. (4)

   c) Evaluate $\operatorname{Res}_{z=\pi i} \dfrac{z - \sinh z}{z^2 \sinh z}$. (2)

3. a) Find the radius of convergence and the circle of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{\sqrt{n}}. \qquad (2)$$

b) Find the points where the function

$$f(z)=\frac{\log(z+4)}{z^2+i} \text{ is not analytic.} \qquad (2)$$

c) Consider $f(z)=z^2-z$ and closed circular region $R=\{z:|z|\le 1\}$. Find points in R where $|f(z)|$ has its maximum and minimum values. (3)

d) If $f(a)=\int_{|z|=4}\frac{z^2+3z-7}{(z-a)^2}\,dz$ for $|a|\neq 4$, determine $f(a)$. (3)

4. a) Let $f(z)=1+z^2+z^4+z^6+\cdots$, z defined on unit disc $|z|<1$, and $\{a_n\}$ be a sequence of real numbers such that $f(z)=\sum_{n=0}^{\infty}a_n(z-5)^n$. Find radius of convergence of the series $f(z)=\sum_{n=0}^{\infty}a_n\, z^n$. (5)

b) What are the zeros of $\sqrt{z}$ (if any)? (2)

c) With the aid of series, prove that the function $f(z)$ defined as follows is entire.

$$f(z)=\begin{cases}\dfrac{\sin z}{z}, & z\neq 0\\ 1, & z=0\end{cases}. \qquad (3)$$

5. a) Show that the Taylor series $\sum_{n=1}^{\infty}n^{-1}(z-3)^n$ converges to $-Log(4-z)$ for $|z=3|<1$. (3)

b) Show that $I=\int_{|z|=r}|z-r|\,|dz|=8r^2$. (2)

c) Find the Cauchy principal value of the integral $\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$. (5)

6. a) Classify the singular points of $f(z)=\frac{z(z-\pi)^2}{[\sin z]^2}$. (2)

b) Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written as $\frac{1}{2i}\int_C \bar{z}\,dz$. (4)

c) Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points $z = 0,\ z = 2$ and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values. (4)

7. a) Let a denote a real number where $-1 < a < 1$, derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty).$$

Hence, show that

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \text{ and } \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}. \quad (5)$$

b) Show that $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. (5)

8. a) Find the image of the half plane $y > 1$ under the transformation $w = (1-i)z$. (3)

b) Find the linear fractional transformation that maps the points $z_1 = -i,\ z_2 = 0,\ z_3 = i$ onto the points $w_1 = -1,\ w_2 = i,\ w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed? (3)

c) Show that a composition of two linear fractional transformations is again a linear fractional transformation. (4)

9. a) Show that the angle of rotation at a non-zero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n (n = 1, 2, \ldots)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point. (3)

b) Show that for any θ and z_0 with $I_m(z_0) > 0$, the mapping $w = e^{i\theta}(z - z_0)/(z - \bar{z}_0)$ is a conformal mapping of $I_m(z) > 0$ onto $|w| < 1$. (3)

c) Under the mapping $w = f(z) = e^{-z}$, find the image in the w-plane of the rectangle $R: 0 < a \le x \le b,\ 0 < c \le y \le d < 2\pi$ in the z-plane. Is the mapping conformal? (4)