BMTC-134

ASSIGNMENT BOOKLET

Bachelor's of Sciences/Arts (General) Programme
(BSCG / BAG)

ALGEBRA

Valid from 1st Jan, 2023 to 31st Dec, 2023

School of Sciences
Indira Gandhi National Open University
Maidan Garhi
New Delhi-110068
(2023)

Dear Student,

Please read the section on assignments in the Programme Guide for B.A./B.Sc. (General) that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment in three parts**, Part A, Part B, Part C. Part A pertains to Blocks 1 and 2 of the course, Part B to Block 3 and Part C to Block 4. The total marks of the three parts are 100, of which **35% are needed to pass it**. All 3 parts are in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully:

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.: ……………………………………………

NAME: ……………………………………………

ADDRESS: ……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE: …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE: …………………………..….. **DATE:** ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) Solve Part A, Part B and Part C of this assignment **separately**, and **submit the complete assignment answer sheets within the due date.**

6) The assignment answer sheets are to be submitted to your Study Centre within the due date. **Answer sheets received after the due date shall not be accepted.**

   We strongly suggest that you retain a copy of your answer sheets.

7) This assignment is **valid from 1st Jan, 2023 to 31st Dec, 2023**. If you have failed in this assignment or fail to submit it by Dec, 2023, then you need to get the assignment for the year 2024, and submit it as per the instructions given in the Programme Guide.

8) **You cannot fill the examination form for this course** until you have submitted this assignment.

We wish you good luck.

ASSIGNMENT

Course Code: BMTC-134
Assignment Code: BMTC-134/TMA/2023
Maximum Marks: 100

PART-A (MM: 50 Marks)

(To be done after studying Blocks 1 and 2.)

1. Which of the following statements are true? Give reasons for your answers. (10)

   i) If a group G is isomorphic to one of its proper subgroups, then $G = \mathbb{Z}$.

   ii) If x and y are elements of a non-abelian group $(G, *)$ such that $x * y = y * x$, then $x = e$ or $y = e$, where e is the identity of G with respect to $*$.

   iii) There exists a unique non-abelian group of prime order.

   iv) If $(a, b) \in A \times A$, where A is a group, then $o((a, b)) = o(a)o(b)$.

   v) If H and K are normal subgroups of a group G, then $hk = kh \ \forall \ h \in H, k \in K$.

2. a) Prove that every non-trivial subgroup of a cyclic group has finite index. Hence prove that $(\mathbb{Q}, +)$ is not cyclic. (7)

   b) Let G be an infinite group such that for any non-trivial subgroup H of G, $|G : H| < \infty$. Then prove that

      i) $H \leq G \Rightarrow H = \{e\}$ or H is infinite;

      ii) If $g \in G, g \neq e$, then $o(g)$ is infinite. (5)

   c) Prove that a cyclic group with only one generator can have at most 2 elements. (3)

3. a) Using Cayley's theorem, find the permutation group to which a cyclic group of order 12 is isomorphic. (4)

   b) Let τ be a fixed odd permutation in S_{10}. Show that every odd permutation in S_{10} is a product of τ and some permutation in A_{10}. (2)

   c) List two distinct cosets of $< r >$ in D_{10}, where r is a reflection in D_{10}. (2)

   d) Give the smallest $n \in \mathbb{N}$ for which A_n is non-abelian. Justify your answer. (2)

4. Use the Fundamental Theorem of Homomorphism for Groups to prove the following theorem, which is called the **Zassenhaus (Butterfly) Lemma:**

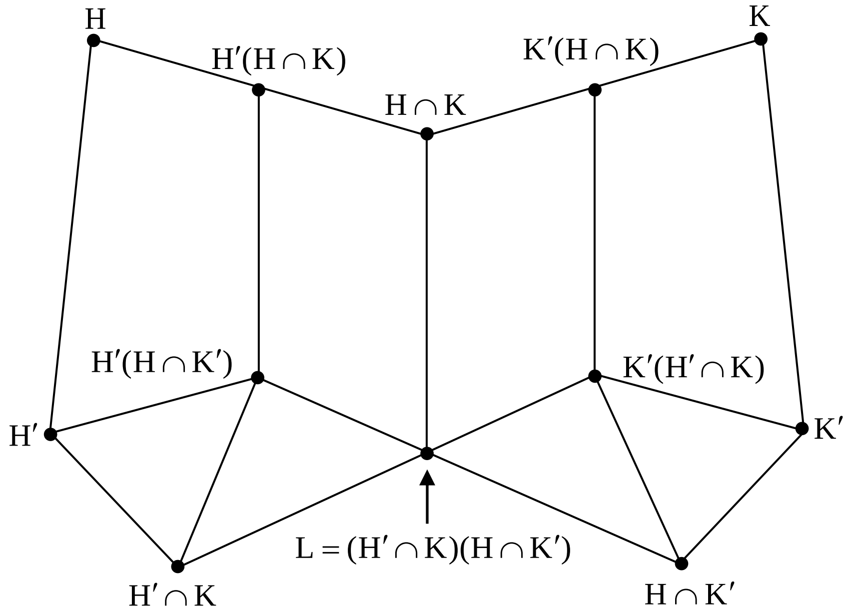
   Let H and K be subgroups of a group G and H' and K' be normal subgroups of H and K, respectively. Then

   i) $H'(H \cap K') \lhd H'(H \cap K)$

   ii) $K'(H' \cap K) \lhd K'(H \cap K)$

iii) $\dfrac{H'(H\cap K)}{H'(H\cap K')} \simeq \dfrac{K'(H\cap K)}{K'(H'\cap K)} \simeq \dfrac{(H\cap K)}{(H'\cap K)(H\cap K')}$ (15)

The situation can be represented by the subgroup diagram below, which explains the name 'butterfly'.



PART-B (MM: 30 Marks)

(Based on Block 3.)

5. Which of the following statements are true, and which are false? Give reasons for your answers. (10)

   i) For any ring R and $a, b \in R$, $(a+b)^2 = a^2 + 2ab + b^2$.

   ii) Every ring has at least two elements.

   iii) If R is a ring with identity and I is an ideal of R, then the identity of R/I is the same as the identity of R.

   iv) If $f : R \to S$ is a ring homomorphism, then it is a group homomorphism from $(R, +)$ to $(S, +)$.

   v) If R is a ring, then any ring homomorphism from $R \times R$ into R is surjective.

6. a) For an ideal I of a commutative ring R, define $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$. Show that

   i) $\sqrt{I}$ is an ideal of R.

   ii) $I \subseteq \sqrt{I}$.

   iii) $I \neq \sqrt{I}$ in some cases. (5)

   b) Is $\dfrac{R}{I} \times \dfrac{R}{J} \simeq \dfrac{R \times R}{I \times J}$, for any two ideals I and J of a ring R? Give reasons for your answer. (5)

7. Let S be a set, R a ring and f be a 1-1 mapping of S onto R. Define + and $\cdot$ on S by:

$x + y = f^{-1}(f(x)) + f(y))$

$x \cdot y = f^{-1}(f(x) \cdot f(y))$

$\forall\ x,\ y \in S.$

Show that $(S, +, \cdot)$ is a ring isomorphic to R. (10)

PART-C (MM: 20 Marks)

(Based on Block 4.)

8. Which of the following statements are true, and which are false? Give reasons for your answers. (10)

   i) If k is a field, then so is $k \times k$.

   ii) If R is an integral domain and I is an ideal of R, then Char $(R) =$ Char (R/I).

   iii) In a domain, every prime ideal is a maximal ideal.

   iv) If R is a ring with zero divisors, and S is a subring of R, then S has zero divisors.

   v) If R is a ring and $f(x) \in R[x]$ is of degree $n \in \mathbb{N}$, then $f(x)$ has exactly n roots in R.

9. a) Find all the units of $\mathbb{Z}[\sqrt{-7}]$. (4)

   b) Check whether or not $\mathbb{Q}[x] \big/ < 8x^3 + 6x^2 - 9x + 24 >$ is a field. (3)

   c) Construct a field with 125 elements. (3)