BMTC-133

ASSIGNMENT BOOKLET

Bachelor's Degree Programme

(BSCG / BAG)
REAL ANALYSIS

Valid from 1st January, 2023 to 31st December, 2023

School of Sciences
Indira Gandhi National Open University
Maidan Garhi
New Delhi-110068
(2023)

Dear Student,

Please read the section on assignments in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment** for this course. The assignment is in this booklet.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully:

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE : …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE : ………………………..….. DATE :……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Study Centre as per the schedule made by the study centre. **Answer sheets received after the due date shall not be accepted.**

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2023. If you have failed in this assignment or fail to submit it by December, 2023, then you need to get the assignment for the year 2024 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the Exam Form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment

Course Code: BMTC-133
Assignment Code: BMTC-133/TMA/2023
Maximum Marks: 100

1. Which of the following statements are true or false? Give reasons for your answers. (10)

   a) The singleton set $\{x\}$ for any $x \in \boldsymbol{R}$ is an open set.

   (b) The series is $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ is a convergent series.

   c) The function $f(x)=\begin{cases} e^{-x}+e^{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x=0 \end{cases}$ is continuous on $[0, 1]$.

   d) The function f defined by $f(x)=|x-\sqrt{2}|\,\forall x \in \boldsymbol{R}$ has a critical point at $x=\sqrt{2}$.

   e) If a function has finitely many points of discontinuities, then the function is not integrable.

2. a) Prove that the sequence $\{a_n\}$ where $a_n=\frac{2^2}{n^2+3^2}$, converges to 0. (3)

   b) Find the following limit, if it exists:

   $$\lim_{x \to 0} \frac{x^3 \sin x^3}{1-\cos x^3}$$ (2)

   c) Test the convergence of the following series. (6)

   i) $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\cdots$

   ii) $\sum \frac{\sqrt{n^{4+1}}-\sqrt{n^{4-1}}}{n}$

3. a) Explain the order completeness property of $\boldsymbol{R}$, and use it to show that the set $S=\left\{\frac{n}{n+1} \middle| n \in \boldsymbol{N}\right\}$ has a supremum as well as infimum in $\boldsymbol{R}$. (3)

   b) Let f be the function defined by

   $$f(x)=\begin{cases} 2x-1, & \text{if } x \in]\infty, 1[\\ \frac{3x^2-2}{x}, & \text{if } x \in [1, 2[\\ (1+2x)^2, & \text{if } x \in [2, \infty[\end{cases}$$

   Discuss the continuity of f on $]\infty, \infty[$. (4)

   c) Check whether the following sets are open, closed or neither:

   i) $]1, 5[\cup [3, 6]$

   ii) $[0, 1] \cup \left\{\frac{5}{9}, \frac{3}{4}, \frac{10}{7}\right\}$

   iii) $\{5n : n \in \boldsymbol{N}\}$ (3)

4. a) Using the principle of mathematical induction, prove that 7 is a factor of $3^{2n-1} + 2^{n+1}$, $\forall n \in \boldsymbol{N}$. (4)

b) Show that the equation $x^3 - 2x^2 + 5x - 12 = 0$ has a root which is a positive real number. (4)

c) Prove that the set $\left\{\frac{3}{6}, \frac{3}{7}, \frac{3}{8}, \dots\right\}$ is a countable set. (2)

5. a) Show that the local maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$. (4)

b) Verify Cauchy Mean Value Theorem for the functions

$$f(x) = x,\ g(x) = \frac{1}{x},\ x \in [1, 4].$$ (3)

c) Show that $1 + x \le e^x$, $\forall x \in [0, \infty[$. Does the inequality hold for $x < 0$? Justify your answer. (3)

6. a) By showing that the remainder after n-terms tends to zero, find Maclaurin's series expansion of $\sin 2x$. (5)

b) Find the greatest value of the function $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 7$ over the interval $[0, 1]$. (5)

7. a) Consider the function $f(x) = 2\cos x$ in the interval $\left[0, \frac{\pi}{2}\right]$. Show that $L(P_1, f) \le L(P_2, f)$ and $U(P_2, f) \le U(P_1, f)$ where $P_1 = \left\{0, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and $P_2 = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$. (6)

b) Show that the derivative f' of the following function f given by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

exists at $x = 0$ but f' is not continuous at 0. (4)

8. a) Check whether the following function has a mean value in the interval $[2, 5]$

$$f(x) = \begin{cases} 1 & \text{if } 2 \le x < 3 \\ 3 & \text{if } 3 \le x \le 5 \end{cases}$$

Does this contradict the mean value theorem? Justify. (3)

b) Find the limit as $n \to \infty$, of the sum

$$\frac{n}{3n^2 + 1^2} + \frac{n}{3n^2 + 2^2} + \frac{n}{3n^2 + 3^2} + \dots + \frac{1}{4n}.$$ (4)

c) Apply Weierstrass M -test to show that the series $\sum \frac{10}{n^4 + x^4}$ converges uniformly for all $x \in \boldsymbol{R}$. (3)

9. a) Using Riemann integration show that $\int_1^2 (3x+1)dx = \frac{11}{2}$. (5)

b) Show that the function $f(x) = \frac{1}{x}$ is continuous on $]0, 1]$ but not uniformly continuous. (5)

10. a) Give one example for the following. Justify your choice of examples.

i) A bounded set having no limit point.

ii) A bounded set having infinite number of limit points.

iii) A infinite compact set which is not an interval. (6)

b) Prove that the function f defined by

$$f(x) = \begin{cases} 4, & \text{if } x \text{ is rational} \\ -4, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at each real number, using the sequential definition of continuity. (4)