MTE-14

ASSIGNMENT BOOKLET

Bachelor's Degree Programme
(B.Sc./B.A./B.Com.)

MATHEMATICAL MODELLING

(Valid from 1st January, 2023 to 31st December, 2023)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
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Maidan Garhi
New Delhi-110068
(2023)

Dear Student,

Please read the section on assignments in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment** for this course. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully:

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.: ……………………………………………

NAME: ……………………………………………

ADDRESS: ……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE: …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE: ……………………….….. **DATE:** ……………………….……………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) While solving problems, clearly indicate which part of which question is being solved.

6) **This assignment is valid only upto December, 2023**. If you have failed in this assignment or fail to submit it by the last date, then you need to get the assignment for the next cycle and submit it as per the instructions given in that assignment.

7) It is compulsory to submit the assignment before filling in the exam form.

We strongly suggest that you retain a copy of your answer sheets.

We wish you good luck.

ASSIGNMENT

(To be done after studying all the blocks)

Course Code: MTE-14
Assignment Code: MTE-14/TMA/2023
Maximum Marks: 100

1. a) Using dimensional analysis, show that the planets obey Kepler's third law. (3)

   b) A raindrop starts falling from the clouds at a considerable height above the surface of the earth. During the fall, the raindrop experiences retardation due to air resistance, which is directly proportional to the instantaneous speed $v(t)$ of the drop.

   i) Write the model equations
   ii) Is this system static or dynamic? Why?
   iii) Obtain an expression for the speed $v(t)$.
   iv) Discuss the behaviour of $v(t)$ as the time t changes. (3)

   c) A stone is dropped vertically from a tower of height h. At the same time, another stone is
   thrown vertically upwards from the base of the tower with a velocity u. What is the minimum value of u so that the two stones will meet each other mid-air? (4)

2. a) Consider the problem of heat conduction in a one dimensional slab ($0 < x < 1$). The surface at $x = 0$ is insulated and that at $x = 1$ is losing heat to the environment at a rate proportional to the temperature difference between the surface and the surrounding. The formulation of this problem results in the following pde with boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = u_0(x), \ \frac{\partial u}{\partial x}(0,\ t) = 0$$

$$\frac{\partial u}{\partial x}(1,\ 0) + u(1,\ 0) = 0$$

   where $u(x,\ t)$ is the temperature.

   Using the separation of variable method seek the solution $u(x,t) = T(t)X(x)$ and find

   i) equation for $T(t)$, $X(x)$ and corresponding conditions on $X(x)$ to solve these equations
   ii) the eigenvalues and the corresponding eigenfunctions of the bvp obtained in i) above for $X(x)$
   iii) the solution $u(x,t)$. (6)

   b) A cup of coffee at 90° C is kept in a room at 25° C. Two minutes later, the temperature of the coffee is 75° C. When will the temperature of the coffee reach 50° C? (4)

3. a) A patient is given a dose Q mg/ml of a drug at regular interval of time t. The concentration C, of the drug in the blood has been shown experimentally to obey the law

$$\frac{dC}{dt} = -k\ e^{C}$$

   i) if the first does is administered at $t = 0$ hr. then find the concentration after T hr. have elapsed.

ii) assuming an instantaneous rise in concentration whenever the drug is administered, find the concentration after the second dose and T hr. have elapsed again.

iii) show that the limiting value R of the concentration for doses of $Q\,mg/ml$ repeated at interval T hr. is given by the formula

$$R = -\ln \frac{kT}{1-e^{-Q}}$$ (6)

b) Discuss the static stability and dynamic stability for the following demand and supply functions where we assume $k = 3$.

$$D_t = -0.2\,p_t + 80$$
$$S_t = 0.3\,p_t + 40$$ (4)

4. a) A spherical cell of radius a is taking in a nutrient from its surroundings and metabolizing it. Assume that the concentration of nutrient in the cell at $r = a$ is zero, for $t > 0$, and the initial concentration is C_0 for $r < a$. Solve the following differential equation corresponding to the model of this problem:

$$D\left(\frac{\partial^2 C}{\partial r^2} + \frac{2}{r}\frac{\partial C}{\partial r}\right) = \frac{\partial C}{\partial t}$$

Find the concentration $C(r, t)$ at any instant of time. Also write the steady-state solution. (6)

b) A lake is initially stocked with 100 fishes and 1000 zooplanktons on which the fishes prey. There is ample food for the zooplanktons. Because fishes prey on zooplanktons the population of fishes will increase at a rate proportional to the number of encounters between the species. Fishes will also die at a rate proportional to the fish population. Also fishing is permitted at the rate of $\frac{1}{5}$th of the existing population at any time. Zooplanktons multiply at a rate proportional to the population and die off at a rate proportional to the number of encounters between the two species. Model this situation. (4)

5. a) Suppose that a given population can be divided into parts, those who have a given disease and can infect others and those who do not have it, but are susceptibles. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals, then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and the rate of spread is proportional to the number of such contacts. If y_0 is the initial proportion of infectious individuals then

i) Formulate a mathematical model for the given problem and write a differential equation governing it.

ii) Find the equilibrium points of the equation obtained in (i).

iii) Solve the given problem. What happens to the spread of the disease as $t \to \infty$? (6)

b) A company manufactures an item at a rate of 12 items per day (following an exponential distribution). The service time distribution is also exponential with an average of 60 minutes

i) Calculate the utilization factor

ii) Find the average number of items in the queue.

iii) What is the probability that the queue size is greater than or equal to 5? (4)

6. a) The return distribution on 2 securities, A and B is as follows:

Event (j)	Chance $p_{1j} = p_{2j}$	Return	
		R_{1j}	R_{2j}
1	0.33	19	18
2	0.25	17	16
3	0.17	11	11
4	0.25	10	9

Find which security is more risky in the Markowitz sense. (6)

b) Geometrically interpret the growth rate corresponding to the population growth model

$$\frac{dx}{dt} = r_1 x\left(\frac{x}{k_0} - 1\right)\left(1 - \frac{x}{k}\right), \quad 0 < k_0 < k$$

$x(0) = x_0$,

where r_1, k, k_0 are constants, k being the carrying capacity of the population $x(t)$. Hence find when the growth is maximum. (4)

7. a) Consider the cubic total cost function

$$C = 0.004q^3 - 0.8q^2 + 10q + 5$$

Assume that the price of q is 13 per unit. Find the output which yields maximum profit. (5)

b) Consider the pay-off table for two players given below

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 0 & 5 & 2 \\ -2 & 4 & 3 \\ 2 & -3 & -4 \end{bmatrix} \end{array}$$

By graphical procedure determine the value of the game and the optimal mixed strategy for each player. (5)

8. a) Suppose that the previous forecast was 2083 and the actual value of the variable of interest for the last period was 1975 and the oldest value of interest was 1945. Using the moving average technique based upon the most recent four observations find new forecast for the next period. (4)

b) Let the utility function of a consumer be $U = q_1\sqrt{q_2}$. Let $p_1 = \text{Rs.}250$, $p_2 = \text{Rs.}400$ and that the consumers income for the period is Rs.15000. Obtain the quantities required by the consumer so that his utility function is maximized by consuming this combination. (3)

c) There are N identical firms producing a particular commodity. The cost function for each firm producing q units is $q^3 + 2q^2 + 4q + 6$ units of money. Obtain the supply function for each firm. The demand function is $D(p) = N\left(\frac{22}{3} - p\right)$. Find the equilibrium price. (3)

9. a) The sales of a company from 1993-1998 are given below

Year	1993	1994	1995	1996	1997	1998
Sales (in lakhs of rupees)	40	45	50	55	60	65

Fit a linear curve using the least squares method. Hence find out the company's sales in 1999. (4)

b) Let $P = (w_1, w_2)$ be a portfolio of two securities. Find the value of w_1 and w_2 in the following situations

i) $\rho_{12} = -1$ and P is risk-free.

ii) $\sigma_1 = \sigma_2$ and variance P is minimum.

iii) Variance on P is minimum and $\rho_{12} = -0.5$, $\sigma_1 = 2$ and $\sigma_2 = 3$. (6)

10. a) Television sets for repair arrive at random at an average rate of 4 per day to a single repairman who takes an average of $1\frac{1}{2}$ hours to carry out each repair. It being assumed that the repair times have an exponential distribution. What is the average number of television sets in the workshop? What is the probability that an arriving set will find at least 3 sets in front of it? The repairman works for eight hours a day.

(3)

b) State the type of modeling you will use for the following problems, giving reasons for your answers. Also state four-essentials for these problems.

i) The economic viability of an insurance company depends critically on its ability to assess risks and decide on the premium charged to cover risks. If the premium are low, then payouts can exceed revenue collected and the company can go bankrupt. On the other hand, if they are high, the number of customers will go down, thus affecting profitability. To help the insurance company decide the premium it should charge for different risks to ensure economic viability and maximize its profits.

ii) Companies located on the banks of a river and producing chemicals dispose their waste by indiscriminately dumping it into the river, causing high levels of pollution. Local authorities passed new legislation with very high fines if the pollution in the river exceeded certain specified concentration limits. To find a policy for discharging the waste so as to ensure that the concentration level never exceed the specified limits. (7)