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00338

MASTER'S IN MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE M.Sc. (MACS)

Term-End Examination
June, 2010

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Question no. 1 is* ***compulsory****. Do* ***any four*** *questions out of questions no. 2 to 7. Calculators are* ***not*** *allowed.*

1. State, whether the following statements are ***True*** or ***False***. Give reasons to justify your answer : **2x5=10**

(a) The set $F := \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ is a closed set in **R** with the discrete metric on it.

(b) Every Lehesgne integrable function on **R** is Riemann integrable.

(c) Every convergent sequence in a metric space is a bounded sequence.

(d) If $A = \{(x, y) \epsilon \ \mathbf{R}^2 : x^2 + y^2 = 4\}$ then $\text{int}(A) = \phi$.

(e) Function $f : \mathbf{R} \to \mathbf{R}^2$ defined by $f(x) = (f_1(x), f_2(x))$

Where $f_1(x) = x$, $f_2(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0), \end{cases}$

is not differentiable at $x = 0$.

2. (a) State Faton's lemma. Show by an example that the inequality in the lemma cannot be replaced by equality. 3

(b) Show that the system R_1 given by 4

$g(t) = (R_1 f)(t) = t f(t)$

is a time - varying system whereas the system R_2 given by

$g(t) = (R_2 f)(t) = \int_{-\infty}^{t} f(t)\,dt$ is a time invariant system.

(c) Use the dominated convergence theorem to find 3

$$\lim_{n \to \infty} \int_1^{\infty} fn(x)\,dx$$

where $fn(x) = \dfrac{nx}{1+n^2x^2}$

3. (a) Let $\{F_n\}$ be a sequence of non empty closed subsets of a complete metric space X such that $F_n \supseteq F_{n+1}$ for each positive integer n and $d(F_n) \to 0$. Let $F = \bigcap_{n=1}^{\infty} F_n$. Then, prove that, F is a singleton set. Show by an example that the conclusion does not hold if X is not complete. 6

(b) If f is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$, then prove that, $\mathbf{f}'' = 0$. 4

4. (a) Define the outer measure m* of any set $A \subseteq \mathbf{R}$. For any subsets A_1 :, A_2 of **R**, show that $m^* (A_1 \cup A_2) \subseteq m^* (A_1) + m^* (A_2)$. 5

(b) Compute the Fourier series of the function g given by. 5

$$g(t) = \begin{cases} -2 \ , & -\pi < t < 0 \\ 2 \ , & 0 < t < \pi \end{cases}$$

5. (a) Define the interior, closure and the boundary of any subset of a metric space. Find the interior, closure and boundary of the subset B of $\mathbf{R}^2$ where, 6

$B = \{(x, y) \in \mathbf{R}^2 : x = 0\}$.

(b) State the Inverse function theorem for vector-valued functions. Show that the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ defined by 4

$f(x, y, z, w) = (x + 2y, x^2 - y^2, wz, y + w)$ is locally invertible at the point $(0, 1, -1, 2)$.

6. (a) Let {fn} be a sequence of measurable functions on **R** such that $\sum_{n=1}^{\infty} \int |fn| \, dm < \infty$. 4

Show that the series $\sum_{n=1}^{\infty} fn(x)$ converges for almost all x and its sum is integrable. Further show that

$$\int \sum_{n=1}^{\infty} fn \, dm = \sum_{n=1}^{\infty} \int fn \, dm$$

(b) If a set E has finite measure (i.e. $m(E) < \infty$) then, show that, $L^2(E) \subset L^1(E)$. 3

(c) Let $X = \mathbf{R}^2$ with the discrete metric d. Then, show that, the sequence $\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ does not converge to (0, 0). 3

7. (a) Let A and B be two disjoint non-empty closed sets of a metric space X. Prove that there exists a continuous function $f : X \to \mathbf{R}$ such that $0 \le f \le 1$, $f = 0$ on A and $f = 1$ on B. 5

(b) Show that a convex set in $\mathbf{R}^n$ is path connected. 2

(c) Give an open cover for $\mathbf{R}$ which does not have a finite subcover. What can you say about the compactness of $\mathbf{R}$? Justify your answer. 3
