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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

00728

Term-End Examination

December, 2011

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

Note : *Question number 8 is **compulsory**. Answer any **six** questions from question number **1 to 7**. Use of calculator is **not** allowed.*

1. (a) Two random variables X and Y have the following joint p.d.f. **9**

$$f(x,y)=\frac{3}{16}(x^2+y^2)\,;0<y<x\leq 2$$

(i) Find the marginal distribution of y.

(ii) Conditional expectation of Y given $X=x$

(iii) $P\left[\frac{1}{2}<Y<1\,\middle|\,\frac{1}{2}<X<\frac{3}{2}\right]$

(b) Explain Markov chain with the help of an example state and derive chapman-kolomogorov equation. **6**

2. (a) A Markov chain has an initial distribution $u^{(0)} = \left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ and has the following transition matrix : **7**

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

(i) Check whether this is irreducible and a periodic.

(ii) Find its stationary distribution. Is it unique ? Justify.

(iii) Verify that limiting distribution of the chain is stationary.

(b) The arrivals at a counter in a bank occur in accordance with a Poisson process at an average rate of 8 per hour. The duration of service of a customer has exponential distribution with a mean of 6 minutes. Find the probability that an arriving customer **6**

(i) has to wait on arrival,

(ii) finds 4 customers in the system,

(iii) has to spend less than 15 minutes in the bank.

Also, estimate the fraction of the total time that the counter is busy.

(c) Prove that the processes $\{N_t, t \geq 0\}$ and $\{N_t + X, -1, t \geq 0\}$ have same probability law. **2**

3. (a) Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates of servers 1 and 2 are 8 and 10 respectively. A customer on completion of service at server 1 is equally likely to go to server 2 to leave the system (i.e., $P_{11} = 0$, $P_{12} = 1/2$) whereas a departure from server 2 will go to server ⊥ with 25 percent of the time and will depart the system otherwise (i.e. $P_{21} = 1/4$, $P_{22} = 0$). Determine the limiting probabilities, expected length of queue and expected waiting time of a customer in the system. 9

(b) Let $\{X_n\}$ be a branching process, where the probability distribution of the numbers of offspring is geometric, with $p_n = P[\xi = n] = qp^n$, $n = 0, 1, 2...$ and $q = 1 - p$. Then, find the probability generating function of $\{X_n\}$. 6

4. (a) Describe $M | M | K | \infty$ queueing system, stating the assumptions. The arrival of customers follows Poisson distribution and service time has an exponential distribution. Derive the expression for probability that system will be idle. 9

(b) What do you mean by spectral decomposition of a matrix A ? Obtain the spectral decomposition of 6

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

5. (a) A bag contains 3 white and 5 black balls. 4 balls are transferred to another empty bag. From this bag, now a ball is drawn and is found to be white. What is the probability that out of four balls transferred, 3 are white and 1 is black ? **5**

(b) Consider the covariance matrix for random vector $[X_1, X_2]$ as given below : **10**

$$\Sigma = \begin{bmatrix} 1 & 4 \\ 4 & 100 \end{bmatrix}$$

obtain the principle components using Σ. Obtain the correlation matrix R associated with Σ and show that principle components obtained from R are different from those obtained from Σ. Find the proportion of total population variance explained by the components and interpret it.

6. (a) Two samples of size 100 bars and 160 bars were taken from the lots produced by method 1 and method 2. Two characteristics X_1 = lather and X_2 = mildness were measured. The summary statistics for bars produced by methods 1 and 2 is given by **10**

$$\bar{X}_1 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\bar{X}_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

Test at 5% level of significance whether $\mu_1 = \mu_2$ or not.

[You may like to use the following values $F_{2,120} = 3.07$, $F_{2,\infty} = 3.00$, $F_{3,120} = 2.68$.]

(b) Let $y \sim N_3(\mu, \Sigma)$, where $\mu = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$ and 5

$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}$. Obtain the distribution of

$z = cy$ where $c = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix}$.

7. (a) What do you mean by cannonical correlation ? Explain various steps of cannonical correlation analysis if data on N objects for two sets of variables say Set I X_1, X_2, X_3 and Set II Y_1, Y_2 are available. 8

(b) Let X be a random vector with covariance matrix . 7

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 17 & 27 \\ 6 & 27 & 70 \end{bmatrix}$$

Find a lower triangular matrix B and its inverse such that the components of $Y = B^{-1} X$ are uncorrelated with each other having variance unity.

8. State whether the following statements are true or false. Justify your answer : **2x5=10**

(a) Any finite aperiodic irreducible chain is necessarily ergodic having stationary distribution.

(b) Let $\{\hat{N}_t : t > 0\}$ be defined as $\hat{N}_0 = 0$ and $\hat{N}_t = \text{Sup}\,\{k : S_k \leq t\}$, then $\hat{N}$ coincides with renewal process N.

(c) If a queueing system is represented by $E_3 | E_1 | 1$, it means the interarrival time is exponential and queue discipline is LCFS.

(d) If A is an idempotent matrix and P is any singular matrix, then $PAP^{-1} = B$ is also an idempotent matrix.

(e) Cannonical correlation is a particular case of multiple correlation.