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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00768

Term-End Examination

June, 2015

MMT-004 : REAL ANALYSIS

Time : 2 hours

Maximum Marks : 50

(Weightage : 70%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *questions from questions no.* ***2*** *to* ***7****. Calculators are* ***not*** *allowed.*

1. State whether the following statements are *True* or *False*. Give reasons for your answer. *5×2=10*

   (a) Every Cauchy sequence in a metric space is convergent.

   (b) The set of all non-zero real numbers with usual metric is connected.

   (c) The point (0, 0, 0) is a stationary point for the following function :

   $f : \mathbf{R}^3 \to \mathbf{R}$ defined by

   $f(x, y, z) = (y - x^2)(y - 2x^2) + z.$

(d) The function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in Q \\ 0, & \text{if } x \notin Q \end{cases}$$, where Q is the set of all rational numbers, is Lebesgue Integrable.

(e) Every infinite set in a metric space has a limit point.

2. (a) State the Inverse Function theorem for $\mathbf{R}^n$. Use the Inverse Function theorem to check the local invertibility of the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$f(x, y) = (x^2 - y^2, 2xy).$$ 4

(b) Let d and D be metrics defined on $\mathbf{R} \times \mathbf{R}$ as

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

$$D((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Show that d and D are equivalent metrics on $\mathbf{R} \times \mathbf{R}$. 3

(c) Prove that the outer measure of the interval [3, 5] is 2. 3

3. (a) Define a path connected set in a metric space.

Let $B = \{(x, y) \mid 0 \le d(x, y) \le 2\}$.

Find a path from

(i) $(-1, 0)$ to $(1, 0)$

(ii) $(-1, \frac{1}{2})$ to $(1, \frac{1}{2})$.

Also give an example of a set which is not path connected set. *5*

(b) Check whether the following function is Lebesgue measurable or not :

$$f(x) = \begin{cases} x^2, & \text{if} \quad x \in Q \\ -x^2, & \text{if} \quad x \notin Q, \end{cases}$$

where Q is the set of all rational numbers. *2*

(c) Find the convolution $f * g$ of the following function : *3*

$$f(t) = \frac{t^{1/3}}{1-2(1-t)^{1/3}}, \ 0 < t < 1$$

$$g(t) = 1 - 2t^{1/3}, \ 0 < t < 1.$$

4. (a) If a set E has finite measure, then show that $L^2(E) \subset L^1(E)$. *3*

(b) Let (X, d_1), (Y, d_2) be two metric spaces and $f : X \to Y$. When is f said to be continuous at $x_0 \in X$? Show that f is continuous at x_0, if and only if for every sequence $\{x_n\}$ in X converging to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$ in Y. *4*

(c) Find the extreme values of the function

$$z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

subject to the constraint

$$x_1 + x_2 + x_3 = 20; x_1, x_2, x_3, \geq 0.$$ *3*

5. (a) Show that if $f : (X, d_1) \to (Y, d_2)$, where X is compact is continuous, then f is uniformly continuous. *5*

(b) Let $\{f_n\}$ be a monotonically increasing sequence of non-negative measurable functions defined on a measurable set E. Let $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in E$. Show that $\lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm$. *5*

6. (a) State Fatou's Lemma. Show by an example that the inequality in the lemma cannot be replaced by equality. *3*

(b) Let $X = \mathbf{R}^2$ with the discrete metric d. Then show that the sequence $\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ does not converge to (0, 0). *3*

(c) Let $f(x, y, z) = x^2 e^y$, $g(x, y, z) = y^2 e^{xz}$ and $\phi = (f, g)$. Find $\phi''(1, 2, 3)$. *4*

7. (a) Define the terms : interior, closure and boundary of any subset of a metric space. Find the interior, boundary and closure of the subset A of $\mathbf{R}^2$, where $A = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$, with the usual metric. *5*

(b) Let the frequency response H(iw) be given by $H(iw) = \begin{cases} 1, & -W < w < W \\ 0, & \text{otherwise.} \end{cases}$

Find the system response to the signal $f(t) = e^{i\frac{w}{7}t} + 4e^{i\frac{w}{5}t} + 7e^{i2wt}$. *3*

(c) Find the Fourier cosine series for

$f(t) = t, 0 < t < 2$. *2*