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# M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

# [M. Sc. (MACS)]

## Term-End Examination

## June, 2023

### MMT-006 : FUNCTIONAL ANALYSIS

*Time : 2 Hours* *Maximum Marks : 50*

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***Note :*** *(i) Question No. **1** is compulsory.*

*(ii) Answer any **four** questions from Question Nos. **2** to **6**.*

*(iii) Notations are as in the study material.*

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1. Are the following statements True *or* False ? Justify your answers : 5×2=10

(a) For $x = (a_1, a_2, ...., a_n) \in \mathbf{R}^n$, if a map $\|\cdot\|: \mathbf{R}^n \to \mathbf{R}$ is defined by $\| x \| = \max_{1 \le i \le n} | a_i |$, then $\|\cdot\|$ is a norm on $\mathbf{R}^n$.

P. T. O.

(b) The map $f$ on $C_{00}$ defined as :

$$f(x) = \sum_{n=1}^{\infty} x(n) \text{ for } x = (x(1), x(2), .....) \in C_{00},$$

is a continuous linear functional on $C_{00}$.

(c) The eigen values of a positive self-adjoint operator are always contained in the infinite interval $(0, \infty)$.

(d) Every closed subspace of a reflexive space is reflexive.

(e) The norm limit of a sequence of unitary operators in B(H) is unitary.

2. (a) Define equivalent norms. Prove that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a linear space X are equivalent if and only if there exists positive constants $C_1$ and $C_2$ such that : 3

$$C_1 \| x \|_1 \leq \| x \|_2 \leq C_2 \| x \|_1 \; \forall \, x \in X.$$

(b) Prove that a normed linear space X is a Banach space if and only if every absolutely convergent series of elements of X is convergent in X. 4

(c) Define the spectrum of a bounded linear operator on a normed space X. Let $X = C_{00}$ and $A : X \to X$ be given by $Ax(1) = 0, Ax(i) = x(i-1)$, $i = 1, 2, \ldots\ldots$. Prove that $\sigma(A)$ is non-compact. 3

3. (a) State open mapping theorem. Show that the theorem may not hold if one of the normed linear space are not Banach. 5

(b) Prove that the dual of a Hilbert space is also a Hilbert space. 5

4. (a) Given $n \in N$, find a bounded linear functional $f$ on $l^2$ such that $f(e_n) = n$ and $\| f \| = n$. 3

(b) State the Hahn-Banach separation theorem. 3

(c) Let X be a normed space over **C**. $f$ is a complex linear functional on X. Prove that Ref is a real linear functional on X. Prove that $\| \text{Ref} \| = \| f \|$. 4

5. (a) Prove that the space $l^p$ with the $p$-norm is an inner product space if and only if $p = 2$.

3

(b) Is the Riesz-Fischer theorem true in $(C_{00}, \| \cdot \|_2)$ ? Justify your answer. 3

(c) Give *three* examples of self-adjoint, normal and unitary operators, respectively, on a Hilbert space, where the normal operator is neither self-adjoint nor unitary. 4

6. (a) Let $E_1$ and $E_2$ be subsets of a normed linear space X. Show that $E_1 + E_2$ is compact if $E_1$ and $E_2$ are compact. Is $E_1 + E_2$ closed if $E_1$ is compact and $E_2$ is closed ? Justify. 5

(b) Prove that if A is a positive operator on a Hilbert space H, then I + A is invertible in BL (H). 3

(c) Give an example of a linear map which is not continuous but whose graph is closed. 2