No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2022

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

Note : *Question no.* ***8*** *is* ***compulsory****. Attempt any* ***six*** *questions from questions no. 1 to 7. Use of scientific non-programmable calculator is allowed. Symbols have their usual meanings.*

---

1. (a) Consider the Markov chain having the following transition probability matrix :

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \end{bmatrix} \end{array}$$

(i) Draw the diagram of a Markov chain.

(ii) Classify the states of a Markov chain, i.e., persistent, transient, non-null and a periodic state. Also check the irreducibility of Markov chain.

(iii) Find the closed sets.

(iv) Find the probability of absorption to the closed classes. Also find the mean time up to absorption from transient state 3 to 4. *8*

(b) Determine the parameters of the bivariate normal distribution : *7*

$$f(x, y) = K \exp\left[-\frac{8}{27}\left\{(x-7)^2 - 2(x-7)(y+5) + 4(y+5)^2\right\}\right]$$

Also find the value of K.

2. (a) Suppose that the probability of a dry day (State 0) following a rainy day (State 1) is $\frac{1}{3}$ and the probability of a rainy day following a dry day is $\frac{1}{2}$. Write the transition probability matrix of the above Markov chain.

Given that 1^{st} May is a dry day, then calculate

(i) the probability that 3^{rd} May is also a dry day.

(ii) the stationary probabilities. *6*

(b) Let $\underset{\sim}{X} \sim N_4(\underset{\sim}{\mu}, \Sigma)$ with

$$\underset{\sim}{\mu} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -4 \end{pmatrix} \text{ and } \sum = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 9 & -1 \\ 1 & -1 & -1 & 16 \end{bmatrix}$$

Support $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ are two partitioned subvectors of $\underset{\sim}{X}$ such that $\underset{\sim}{Y}' = (x_1 \ x_3)$ and $\underset{\sim}{Z}' = (x_2 \ x_4)$.

(i) Obtain the marginal distribution of $\underset{\sim}{Y}'$.

(ii) Check the independence of $\underset{\sim}{Y}'$ and $\underset{\sim}{Z}'$.

(iii) Obtain the conditional distribution of $\underset{\sim}{Y}' \mid \underset{\sim}{Z}'$; where $\underset{\sim}{Y}' = (x_1 \ x_2)$, $\underset{\sim}{Z}' = (x_3 \ x_4)$.

(iv) Find E $(\underset{\sim}{Y}' \mid \underset{\sim}{Z}')]$; where $\underset{\sim}{Y}'$ and $\underset{\sim}{Z}'$ are same as in (iii). *9*

3. (a) Suppose that customers arrive at a service counter in accordance with a Poisson process with mean rate 2 per minute. Then obtain the probability that the interval between two successive arrivals is

(i) more than 1 minute.

(ii) 4 minutes or less.

(iii) between 1 and 2 minutes. *6*

(b) The body dimensions of a certain species have been recorded. The information of body length L and body weight W are given below :

Body length L (in mm)	Body weight W (in mg)
45	2·9
48	2·4
45	2·8
48	2·9
44	2·4
45	2·3
45	3·1
42	1·7
50	2·4
52	3·7

At 5% level of significance, test the hypothesis that all variances are equal and all covariances are equal in variance-covariance matrix for the given data. *9*

[You may like to use the values, $\chi^2_{9,0\cdot05} = 3\cdot84$, $\chi^2_{10,0\cdot05} = 4\cdot10$, $\chi^2_{11,0\cdot05} = 5\cdot09$]

4. (a) Find the differential equation of pure birth process with $\lambda_K = K\lambda$ and the process start with one individual at time t = 0. Hence, find $p_n(t) = P(N(t) = n)$ [N(t) is the number present at time t] with E (N(t)) and Var (N(t)). Also identify the distribution. *7*

(b) Let $\{X_n; n \geq 1\}$ be an i.i.d. sequence of interoccurrence times with common probability mass function given by

$$P(X_n = 0) = \frac{2}{3}, \; P(X_n = 1) = P(X_n = 2) = \frac{1}{6}.$$

Let $N_t; t \geq 0$ be the corresponding renewal process. Find the Laplace transform $\tilde{M}_t$ of the renewal function, M_t. *4*

(c) Write two advantages and two disadvantages of conjoint analysis. *4*

5. (a) The Tooth Care Hospital provides free dental service to the patients on every Saturday morning. There are 3 dentists on duty, who are equally qualified and experienced. It takes on an average 20 minutes for a patient to get treatment and the actual time taken is known to vary approximately exponentially around this average. The patients arrive according to the Poisson distribution with an average of 6 per hour. The officer of the hospital wants to investigate the following : *9*

(i) The expected number of patients in the queue.

(ii) The average time that a patient spends at the clinic.

(iii) The average percentage idle time for each of the dentists.

(b) For the two-state Markov chain, whose transition probability matrix is

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}; 0 \le p \le 1.$$

Find all stationary distributions. *6*

6. (a) Let p_K, K = 0, 1, 2 be the probability that an individual generates K offsprings. Then find the p.g.f. of $\{p_K\}$. Also calculate the probability of extinction when

(i) $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$ and $p_2 = \frac{1}{2}$.

(ii) $p_0 = \frac{2}{3}$, $p_1 = \frac{1}{6}$ and $p_2 = \frac{1}{6}$. *6*

(b) Let p = 3 and m = 1 and suppose the random variables X_1, X_2 and X_3 have the positive definite covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}4 & 0{\cdot}3 \\ 0{\cdot}4 & 1 & 0{\cdot}2 \\ 0{\cdot}3 & 0{\cdot}2 & 1 \end{bmatrix}$$

Write its factor model. *6*

(c) For X distributed as $N_3(\mu, \Sigma)$, find the distribution of

$$\begin{bmatrix} X_1 & -X_2 & X_3 \\ -X_1 & X_2 & X_3 \end{bmatrix}.$$ *3*

7. (a) The joint density function of random variables X, Y and Z is given as

$$f(x, y, z) = K.x.e^{-(y+z)};$$

$$0 < x < 2, y \geq 0 \text{ and } z \geq 0.$$

Find

(i) the constant K.

(ii) the marginal distributions of X, Y and Z.

(iii) E(X), E(Y) and E(Z).

(iv) the conditional expectation of Y given X and Z.

(v) the correlation coefficient between X and Y. *9*

(b) For the model M|M|1|N|FIFO, calculate the steady state solution for P_0. *6*

E(n) – Average number of customers in the system

E(V) – Average waiting time in the system

8. State which of the following statements are *true* and which are *false*. Give a short proof or a counter example in support of your answer. *10*

(a) For 3 independent events E_1, E_2 and E_3

$$P(E_1 \cup E_2 \cup E_3) + P(\overline{E}_1)\, P(\overline{E}_2)\, P(\overline{E}_3) = 0.$$

(b) The range of multiple and partial correlation coefficient is $(-1, 1)$.

(c) If $\{X(t);\ t \geq 0\}$ is a Poisson process, then $N(t) = [X(t + S_0) - X(t)]$ where $S_0 > 0$ is a fixed constant, is also a Poisson process.

(d) In Hotelling T^2, the value of S is given by

$$S = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \mu)(X_j - \mu)'.$$

(e) Let $\underset{\sim}{X}_{p\times 1} \sim N_p(\mu, \Sigma)$ and $X_{p\times n}$ be the state matrix, then parameters involved in the above distribution are p for μ and $\frac{1}{2}p(p+1)$ for Σ.