MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2022

MMT-004 : REAL ANALYSIS

Time : 2 hours | *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no.* ***1*** *is* ***compulsory****. Attempt any* ***four*** *questions out of questions no. 2 to 6. Use of calculator is* ***not*** *allowed.*

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1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) The function $\varphi(x) = \frac{1}{x}$, $3 \leq x \leq 4$ is not uniformly continuous.

(b) A complete metric space is a countable collection of nowhere dense sets.

(c) The function $f : \mathbf{R} \to \mathbf{R}^2$ given by $f(x, y) = (x, x|x|)$ is differentiable at 0.

(d) Any Lebesgue integrable function is always Riemann integrable.

(e) The image of any connected set in $\mathbf{R}^2$ under the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = x^2 + y^2$ is connected.

2. (a) Let A and B be non-empty disjoint closed subsets of a metric space (X, d). Show that there exist open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \phi$. *3*

(b) Define saddle points. Compute the saddle points of the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = (y - x^2)(y - 2x^2)$. *3*

(c) State the Lebesgue dominated convergence theorem. Find $\lim\limits_{n \to \infty} \int_0^\infty \frac{\sin x}{1 + nx^2}\, dx$. *4*

3. (a) Define components in a metric space. What are all the components of the set of all non-zero real numbers under the

(i) usual metric on $\mathbf{R}$, and

(ii) the discrete metric on $\mathbf{R}$? *3*

(b) Find the directional derivation of the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ defined by

$$f(x, y, z, w) = (x^2y, xyz, x^2 + y^2, zw)$$

at $(1, 2, -1, -2)$ in the direction $\sim = (1, 0, -2, 2)$. *4*

(c) Define measurable sets in **R**. Prove that intervals are measurable. *3*

4. (a) If $f : X \to Y$ is a continuous map between metric spaces X and Y and K is a compact subset of X, then show that f(K) is compact. *3*

(b) Find the Taylor series expansion of the function f given by

$$f(x, y) = x + 2y + xy - x^2 - y^2$$

about the point (1, 1). *4*

(c) Let $f, g \in L'(\mathbf{R})$, define convolution $f * g$ of f and g. Show that if either f or g is bounded, then the convolution $f * g$ exists for all x in **R** and is bounded in **R**. *3*

5. (a) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}$ converges in **R**. *3*

(b) State the second-derivative test for finding the extremum values of a function f using Hessian Hf.

Find the critical points of

$f : \mathbf{R}^3 \to \mathbf{R}$ given by

$f(x, y, z) = x^2 y^2 + z^2 + 2x - 4y + z$

and check whether they are extreme points. *4*

(c) Find the Fourier cosine series expansion of the function

$$f(t) = \begin{cases} 1, & 0 < t \leq \frac{1}{2} \\ 0, & \frac{1}{2} < t < 1 \end{cases}.$$ *3*

6. (a) Show that every path connected metric space is connected. *3*

(b) Consider the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by

$f(x, y, z) = x^2 + y^3 - xy \sin z.$

Prove that the equation $f(x, y, z) = 0$ defines a unique continuously differentiable function of near $(1, -1)$ such that $g(1, -1) = 0$. *3*

(c) Define and give an example for each of the following concepts in the context of signals and systems : *4*

(i) A stable system

(ii) A time-varying system