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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours] *[Maximum : Marks : 50*

Weightate : 70%

Note: Question No. 6 is compulsory. Attempt any four out of question 1 to 5. Notations are same as in the study material.

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1. (a) Use uniform boundedness principle to prove the following:

   Let $\{a_n\}$ be a sequence in K with the property that for every $\{x_n\} \in C_0$ it follows that $\{a_n x_n\} \in C_0$. Then $\{a_n\} \in l^\infty$. 4

   (b) Prove that a closed subspace of a reflexive space is reflexive. 3

   (c) Consider C^3 with respect to the standard innerproduct. Let $V_1 = (1, 1, 0)$, $V_2 = (1, 0, 0)$ and $V_3 = (1, 1, 1)$. Using the Gram-Schmidt orthogonalization process, orthonormalise V_1, V_2 and V_3. 3

2. (a) Define the adjoint of a bounded linear operator acting on a Hilbert space. Find the adjoint of the right shift operator on l^2. 3

(b) Let $X = C[0, 1]$. Show that there is a $T \in BL(X)$ whose spectrum is a given interval $[a, b]$. 4

(c) Prove that the dual of l^∞ contains a proper subspace which is linearly isometric to l^1. 3

3. (a) Suppose Y is the set of all even functions in $C[-1, 1]$. Find the orthogonal complement $Y^\perp$ of Y in $C[-1, 1]$ under the inner product on $C[-1, 1]$ given by $\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)dt$.

($C[-1, 1]$ - the space of all continuous real valued functions on [−1, 1]). 3

(b) Let $k : [0, 1] \times [0, 1] \to C$ be a square integrable function. Define

$T : L^2[0, 1] \to L^2[0, 1]$ as

$$T(f)(t) = \int_0^1 k(t, s) f(s) ds$$

(i) Show that T is a bounded operator on $L^2[0, 1]$.

(ii) Show that T is self-adjoint if $k(s, t) = \overline{k(t, s)}$. $\forall (t, s) \in [0, 1] \times [0, 1]$. 4

(c) Let X and Y be Banach spaces and $T \in BL(X, Y)$. If $R(T)$ is closed in Y, then show that $R(T)$ is linearly homomorphic to $\frac{X}{Z(T)}$. Is the converse true? Justify your answer. 3

4. (a) Prove that a normal linear space X is separable if its dual space X' is separable. Is the converse true? Justify your answer. 6

(b) Define $f : c[-1, 1] \to C$ as

$$f(x) = \int_{-1}^{0} x(t)\,dt - \int_{0}^{1} x(t)\,dt$$

Find $\| f \|$. 4

5. (a) Let X and Y be Banach space and $F : X \to Y$ be a one-one bounded linear map. Prove that its range $R(F)$ is closed in Y if and only if $F^{-1} : R(F) \to X$ is bounded. 4

(b) Find an infinite orthonormal set in l^2. 3

(c) Let X be a normed linear space and $\{x_1, \ldots\ldots\ldots, x_n\}$ be linearly independent in X. Then there exists $f_1, \ldots\ldots\ldots, f_n$ in X' such that:

$f_j(x_i) = \delta_{ij} \quad \forall\ i, j = 1, \ldots\ldots, n$

when $\delta_{ij} = \begin{Bmatrix} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{Bmatrix}$ 3

6. (a) Every bounded linear functional on a normed linear space is compact. 2

(b) The projection map $p : \mathbb{R}^3$ to $\mathbb{R}^3$ given by $p(x_1, x_2, x_3) = (x_1, x_2, 0)$ is an open map. 2

(c) The closed ball $\{x \in l^2 : \|x\|_2 \leq 2\}$ is compact. 2

(d) If $T : X \to Y$ is a continuous linear map, when X and Y are normed linear spaces, then T is uniformly continuous. 2

(e) l^p with the p-norm is an inner product space for all $1 < p < \infty$.

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