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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination, 2019

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours] [Maximum Marks : 100

(Weightage : 50%)

Note : Question No. 8 is compulsory. Attempt **any six** questions from question no. 1 to 7. Use of calculator is not allowed. All the symbols used have their usual meaning.

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1. (a) Let the joint probability density function of two continuous random variables X and Y be

$$f(x,y) = 8xy, 0 < x < y < 1$$
$$= 0 \quad \text{, elsewhere}$$

(i) Find the marginal p.d.f. of X and Y.

(ii) Test independence of X and Y.

(iii) Compute $P[0 < x < 0.4 \mid 0.3 < y < 0.8]$

(iv) Find $V(Y \mid X = x)$. [9]

(b) Let $X \sim N_3(\mu, \sum)$ where $\mu = [213]'$ and

$$\sum = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$ Find the distribution of

$$\begin{bmatrix} X_1 - X_2 + X_3 \\ X_1 + X_2 + 2X_3 \end{bmatrix}.$$ [6]

2. (a) A Markor chain $\{X_n, n=0,1, 2, ...\}$ has initial distribution $u_0 = [0.1, 0.3, 0.6]'$ and transition matrix $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$ having states (1,2,3), obtain :

(i) $P[X_2 = 3]$

(ii) $P[X_1 = 2, X_2 = 3]$

(iii) $P[X_0 = 1, X_1 = 3, X_2 = 2]$ [6]

(b) Determine the principal components, Y_1 and Y_2, for the covariance matrix $\sum = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$. Also calculate the proportion of total population variance for the first principal component. [9]

3. (a) The mean Poisson rate of arrival of planes at an airport during peak hours is 20 per hour. 60 planes per hour can land at the airport in good weather and 30 planes per hour in bad weather in Poisson fashion. Find the following during peak hours :

(i) The average number of planes flying over the field in good weather ;

(ii) The average number of planes flying over the field in bad weather ;

(iii) The average number of planes flying over the field and landing in good weather ;

(iv) The average number of planes flying over the field and landing in bad weather ;

(v) The average landing time in good weather and bad weather. [6]

(b) An equal number of balls are kept in three boxes B_1 , B_2 and B_3. The boxes B_1 , B_2 and B_3 contain respectively 3%, 5% and 2% defective balls. One of the boxes is selected at random and a ball is drawn randomly. If the ball is found to be defective, what is the probability that it has come from B_2?

[4]

(c) Let $Z = [Z^{(1)}, Z^{(2)}]$ and

$$\text{Cov}(Z) = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{21} & \rho_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.4 & 0.5 & 0.6 \\ 0.4 & 1 & 0.3 & 0.4 \\ \hline 0.5 & 0.3 & 1 & 0.2 \\ 0.6 & 0.4 & 0.2 & 1 \end{array}\right]$$

Compute the correlation between the first pair of canonical variates and their component variables.

[5]

4. (a) From the samples of sizes 80 and 100 from two populations, the following summary statistics were obtained :

$$X_1 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, X_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

Where X_1, X_2 are the means and S_1, S_2 are the standard deviations of two populations. Test for the equality of the population means at 5% level of significance. Assume $\sum_1 = \sum_2$. [You may use the following values : $F_{0.05,2,177} = 3.04$, $F_{0.05,2,100} = 3.10$, $F_{0.05,2,80} = 3.15$]. [7]

(b) Describe birth and death processes with the parameter λ. If $\lambda_k = \lambda$ and $\mu_k = k\mu$, $k \geq 0$, λ, $\mu > 0$, then show that the stationary distribution of these process always exists. Obtain the steady state distribution. [8]

5. (a) Let X be $N_3(\mu, \Sigma)$ with $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Examine the independence of the following :

(i) X_1 and X_2 ;

(ii) (X_1, X_2) and X_3 ;

(iii) $X_1 + X_2$ and X_3. [7]

(b) Let { N_n, n= 0,1,2, ...} be a renewal process with sequence of renewal periods $\{X_i\}$. Each X_i follows the binomial distribution with $P[X_i = 0] = 0.6$ and $P[X_i = 1] = 0.4$.

Find the distribution of N_n. [5]

(c) To fit linear are regression on dependent variable Y and independent variables X_1 and X_2 we have the following information :

$E(X_1) = 3$, $E(X_2) = 2$, $\text{Var}(X_1) = 2$, $\text{Var}(X_2) = 1$, $\text{Cov}(X_1, X_2) = 1$, $\text{Cov}(X_1, Y) = 3$, $\text{Cov}(X_2, Y) = 1$, $V(Y) = 9$. Find the multiple correlation coefficient R. [3]

6. (a) A barber shop has two barbers. The customers arrive at a rate of 5 per hour in a Poisson fashion, and the service time of each barber takes an average of 15 minutes according to exponential distribution. The shop has 4 chairs for waiting customers. When a customer arrives in the shop and does not find an empty chair, she leaves the shop. What is the expected number of customers in the shop ? What is the probability that a customer will leave the shop finding no empty chair to wait ? [5]

(b) In a branching process, the offspring distribution (p_k) is given below :

$$p_k = pq^k, q = 1 - p, 0 < p < 1, k = 0, 1, 2, \ldots\ldots$$

What will be the probability of extinction in this branching process ? [5]

(c) Let N_s be a Poisson process with parameter $\lambda > 0$. Fix $s > 0$ and let the renewal function be given by $M_t = N_{(t+s)} - N_s$. Show that the conditional distribution of M_t, given $N_s = 10$, is Poisson. [5]

7. (a) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2 where $X_1 \sim N_2(\mu^{(1)}, \sum)$ and $X_2 \sim N_2(\mu^{(2)}, \sum)$

$$\mu^{(1)} = [1\ 2]' \quad , \quad \mu^{(2)} = [-1\ 0]' \quad \sum = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Considering equal cost and equal prior probabilities, classify the observation [-1 1]' in one of the two populations. [6]

(b) Suppose interoccurrance times $\{X_n : n \geq 1\}$ are uniformily distributed on $[0,1]$:

(i) Find $\overline{M}_t$, the Laplace transform of the renewal function M_t.

(ii) Find $\lim_{t \to \infty} M_t / t$. [4]

(c) Let the rand m vector $X' = [X_1\ X_2\ X_3]$ has

Mean Vector $= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ and

Var-cov matrix $= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix}$. Fit the equation

$Y = b_0 + b_1 X_1 + b_2 X_2$. Also obtain the multiple correlation coefficient between X_3 and $[X_1, X_2]$.[5]

8. State whether the following statements are true or false. Justify your answer with a short proof or a counter example : [10]

(i) If P is a transition matrix of a Markov Chain, then all the rows of $\lim_{n \to \infty} P^n$ are identical.

(ii) Every non-negative definite matrix is a var-cov matrix.

(iii) The multiple correlation coefficient R can lie between -1 and 0.

(iv) Posterior probabilities obtained from Baye's theorem are larger than respective prior probabilities.

(v) If X_1, X_2, X_3 are iid from $N_2(\mu, \sum)$, then $\frac{X_1 + X_2 + X_3}{3}$ follows $N_2(\mu, \frac{1}{3}\sum)$.