01151

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2019

MMT-004 : REAL ANALYSIS

Time : 2 hours

Maximum Marks : 50
(Weightage : 70%)

Note : *(i)* *Question no.* **1** *is* ***compulsory.***

(ii) *Attempt* ***any four*** *questions out of questions no.* **2** *to* **6**.

(iii) *Calculators are* ***not*** *allowed.*

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1. State whether the following statements are **True** or **False**. Give reasons for your answers. **5x2=10**

(a) If (X, d) is a metric space and $f : X \to \mathbf{R}$ is a continuous function, then the set $\{x \in X, f(x) < 1\}$ is open.

(b) Every saddle point of the function $f : \mathbf{R}^2 \to \mathbf{R}$ is an extreme point.

(c) The spaces $L'(E)$ and $L'(E)$ are the same.

(d) Intersection of a countable collection of closed sets is closed.

(e) The Fourier Series of the function $f(x) = |\sin x|$ is a cosine series.

2. (a) Using the definition prove that the interval $[0, 1)$ is not compact. 3

(b) Show that the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ defined by $f(x, y, z, w) = (x^2 + y^2, x + y, wz, yz)$ is locally invertible at the point $(1, 1, 0, 1)$. 3

(c) Use Dominated Convergence Theorem to find 4

$$\lim_{n \to \infty} \int_1^{\infty} \frac{n^2 x^2}{1 + n^4 x^4} \, dx$$

3. (a) Let $X = (\mathbf{R}, d)$ where d is the standard metric and let $F_n = \left\{ x \in \mathbf{R} / 0 \le x \le \frac{1}{n} \right\}$. Verify whether conditions of Cantor's intersection hold for sequence $\{F_n\}$. Does the conclusion also hold ? Justify. 3

(b) Find the partial and total derivatives of the function $f : \mathbf{R}^4 \to \mathbf{R}^2$ defined by $f(x, y, z, w) = (x^3 z, y^2 + w^2)$ at $(2, 2, 2, 2)$. 3

(c) Let $\{b_n\}$ be a sequence of non-negative measurable functions such that $\sum_{j=1}^{\infty} \int |b_j| \, dm$ is finite. Show that the series $\sum_{j=1}^{\infty} b_j(x)$ converges for almost all x to a function f which is integrable and satisfies 4

$$\int \left(\sum_{j=1}^{\infty} b_j \right) dm = \sum_{j=1}^{\infty} \int b_j \, dm$$

4. (a) Show that the function **4**
$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$
Subject to the constraint
$4x_1^2 + x_2^2 + 2x_3^2 = 14,\ x_1, x_2, x_3 \geq 0$
has a minimum value at

$$\left(\frac{81}{100}, \frac{7}{20}, \frac{7}{25}\right).$$

(b) Check the connectedness of the following sets. Justify your answer. **3**
(i) $A = \{(x, y) \in \mathbf{R}^2 : 4x^2 + 9y^2 = 16\}$ in $\mathbf{R}^2$.
(ii) $A = \bigcup A_n$ where $A_n = (n, n+1)$ in $\mathbf{R}$.

(c) Define the outer measure of any set $E \subseteq \mathbf{R}$. Show that the outer measures of the set E and the set $E + y$, where $y \in \mathbf{R}$ are the same. **3**

5. (a) Let (X, d) be any metric space show that $\rho(x, y)$ defined by $\rho(x, y) = \min \{f, d(x, y)\}$ is a metric on X. **3**

(b) Obtain the second Taylor's series expansion of the function given by $f(x_1, x_2) = x_1^2 x_2 + 5x_1 e^{x_2} + e^{x_1} x_2^3$ at (1,0) **4**

(c) Find the Fourier series of the function $f(t) = t^2$ on $[-\pi, \pi]$. **3**

6. (a) Let (X, d) be any metric space and A be a non-empty subset of X. Show that $f(x) = d(x, A)$ is uniformly continuous on X. **3**

(b) Suppose $E \subset \mathbf{R}^n$ is open and $f : E \to \mathbf{R}^m$ is differentiable at a point $a \in E$. Prove that all the partial derivatives of 'f' exist at 'a' and are continuous at 'a'. **4**

(c) Define Translation and Scaling systems and show that they do not commute. **3**
