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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

00225

June, 2018

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is allowed. All the symbols used have their usual meaning.*

1. (a) The random variables, X and Y have the following joint p.d.f. :

$$f(x, y) = \begin{cases} x + y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the

(i) Conditional p.d.f. of X given Y = 0·2.

(ii) Marginal p.d.f. of X.

(iii) $P[X \le 0{\cdot}4 \mid Y = 0{\cdot}2]$. *6*

(b) A single repairperson looks after two machines, 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate λ_i, where $i = 1, 2$. When machine i fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. The repairperson will always serve machine 1 when it is down. For instance, if machine 1 fails while machine 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start on machine 1.

(i) Write down all the states.

(ii) Find all steady state probabilities.

(iii) What is the probability that the machine 2 is down ? *9*

2. (a) Let $Y \sim N_3(\mu, \Sigma)$ where $\mu = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$ and

$$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}.$$

(i) Obtain distribution of $X = CY$ where $C = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

(ii) Obtain $Z = l'Y$ such that $Z \sim N(0, 1)$. *6*

(b) Let N_t be a Poisson process with parameter $\lambda > 0$ and $S_{N(t)}$ be the time of last occurrence before time t and $S_{N(t)+1}$ be the time of next occurrence. Show that the residual time $Y = S_{N(t)+1} - t$ follows exponential distribution with parameter λ. 6

(c) Consider a taxi stand where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, if an arriving customer does not find a taxi waiting, he leaves.

Find

(i) the average number of taxis waiting, and

(ii) the proportion of arriving customers that get taxis. 3

3. (a) Let $\overline{X} = (X_1, X_2, X_3)$ be a random vector and data matrix X be given as

$$X' = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 5 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Obtain

(i) Variance-covariance matrix

(ii) Correlation matrix 10

(b) In a branching process having offspring distribution,

$$p_j = \begin{cases} \frac{1}{4} & j = 0 \\ \frac{1}{4} & j = 1 \\ \frac{1}{2} & j = 2 \end{cases}$$

find the probability of extinction. 5

4. (a) In a village 20% children were suffering from malaria. A blood test ordinarily reports positive in 80% cases if the patient was infected by malaria and in 60% cases reports negative if the patient was not infected by malaria. A child was selected at random from the village and the blood test report was positive. Then what is the probability that he is infected by malaria ? 4

(b) Find the matrix of the following quadratic form and determine its definiteness : 5

$$2x_1^2 + 3x_2^2 + 4x_3^2 + 6x_1x_2$$

(c) A barber shop has two barbers. Customers arrive at a rate of 5 per hour in a Poisson process and service time of each barber takes on average 15 minutes according to exponential distribution. The shop has 4 chairs for waiting customers. When a customer arrives in the shop and does not find an empty chair, he will leave the shop.

(i) What is the expected number of customers in the shop ?

(ii) What is the probability that a customer will leave the shop finding no empty chair to wait ? *6*

5\. (a) Evaluate T^2 for testing $H_0 : \mu' = [5, 6]$ using the following data : *9*

$$X = \begin{bmatrix} 4 & 5 & 4 & 11 \\ 8 & 7 & 9 & 4 \end{bmatrix}$$

(b) Let π_1 and π_2 be two populations having density functions $P_1(x)$ and $P_2(x)$. Suppose the cost of assigning items to π_1 given π_2 in the true population is 15 and the cost of assigning items to π_2 given π_1 in the true population is 20. It is known that 60% of items belong to π_2.

(i) Find the prior probabilities.

(ii) Write the cost of misclassification.

(iii) Determine classification regions.

(iv) If a new item has $P_1(x) = 0{\cdot}6$ and $P_2(x) = 0{\cdot}4$, then in which population will it be assigned ? *6*

6. (a) Let $X \sim N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Examine the independence of

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $X_1 + X_2$ and X_3 *3*

(b) The probability transition matrix of a simple weather model having three states, Sunny (S), Cloudy (C) and Rainy (R) with initial probabilities (0·6, 0·3, 0·1) is given below :

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0{\cdot}5 & 0{\cdot}3 & 0{\cdot}2 \\ 0{\cdot}4 & 0{\cdot}3 & 0{\cdot}3 \\ 0{\cdot}2 & 0{\cdot}5 & 0{\cdot}3 \end{bmatrix} \end{array}$$

(i) Draw the directed graph for the transition matrix.

(ii) Find the probability that three successive days will be sunny starting from the initial day. *6*

(c) Let there be three random variables X_1, X_2 and X_3 with the following covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}6 & 0{\cdot}4 \\ 0{\cdot}6 & 1 & 0{\cdot}3 \\ 0{\cdot}4 & 0{\cdot}3 & 1 \end{bmatrix}$$

Take one underlying factor and write its factor model. 6

7. (a) Determine the nature of states from the following transition matrix of a Markov chain : 8

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}75 & 0 & 0{\cdot}25 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) For a Poisson process $\{X(t) : t \geq 0\}$, define processes $N(t) = X(t + s_0) - X(t)$ and $U(t) = X(t + s_0) - X(s_0)$. Examine which of the processes N(t) and U(t) is a Poisson process. 7

8. State whether the following statements are *True* or *False*. Justify your answers. *5×2=10*

(a) Transition probabilities $p_{ij}^{(m, n)}$ in a time homogeneous Markov chain depend on specific times m, n.

(b) In a branching process, if the offspring probability $p_0 = 0$, then there will never occur an extinction of the process.

(c) If $P(B) > 0$, then $P(A|B)$ is smaller than $P(A \cap B)$.

(d) If A is a positive definite matrix, then $|A| > 0$.

(e) The principal components of a set of variables do not depend upon the scales of the variables.
