MMT-007

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00685

Term-End Examination

June, 2018

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 hours *Maximum Marks : 50*
(Weightage : 50%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *questions out of questions no. 2 to 7. Use of non-programmable scientific calculator is allowed.*

1. State whether the following statements are *True* or *False*. Justify your answers with the help of a short proof or a counter example. *5×2=10*

   (a) For the Legendre polynomial $P_n(x)$ of degree n,

   $$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{1}{n+1}.$$

(b) If $\mathcal{L}$ denotes Laplace Transform, then

$$\mathcal{L}\,[t^n] = \frac{\sqrt{n+1}}{s^{n+1}}, \quad n > -1, \quad s > 0.$$

(c) The heat conduction equation $u_t = u_{xx}$ is approximated by

$$\left(u_m^{n+1} - u_m^{n-1}\right) = \frac{k}{h^2}\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right);$$

then the truncation error is of the order $k^3 + kh^2$.

(d) The PDE $\frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x\,\partial y} + (1 + y^2)\frac{\partial^2 u}{\partial y^2} = 0$

is parabolic in the region $x^2 - y^2 > 1$.

(e) The fourth order Runge-Kutta method is used to solve the initial value problems

$$y' = -200\,y, \quad y(0) = 1.$$

The value of h, so that the method produces stable result is $h < 0{\cdot}015$.

2. (a) Construct Green's function for the differential equation

$$xy'' + y' = 0, \quad 0 < x < l$$

under the condition that y(0) is bounded and y(1) = 0. 5

(b) If $P_n(x)$ is a Legendre polynomial of degree n, then using orthogonality property, show that 5

$$\int_{-1}^{+1} x^2 P_{n+1}(x) P_{n-1}(x) \, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

3. (a) Find the solution of the initial boundary value problem

$$u_t = u_{xx}, \quad 0 \le x \le 1,$$
$$u(x, 0) = \sin(2\pi x), \quad 0 \le x \le 1$$
$$u(0, t) = 0 = u(1, t)$$

Using the Crank-Nicolson method with $\lambda = 0{\cdot}6$ and assuming $h = \frac{1}{3}$, integrate for one time level. 4

(b) Solve the initial value problem

$$y' = x + y^2, \quad y(0) = 1,$$

on the interval (0, 0·4) using the Runge-Kutta second order method with h = 0·2. 4

(c) Find the Fourier sine transform of the following function : 2

$$f(x) = \begin{cases} \pi x & , \quad 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

4. (a) Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = \frac{t^n}{\lfloor n} e^{at}.$$ 3

(b) Using the Frobenius method, find the power series solution of the equation

$$(x^2 - x)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$$

about $x = 0$. 5

(c) For single-step methods, what is the numerical error at a nodal point ? When is the numerical single-step method said to be stable ? 2

5. (a) Using Laplace transforms, solve

$$\frac{d^2y}{dt^2} + 9y = t$$

given that $y(0) = 0$, $y(\pi/2) = 0$. 4

(b) Using the five-point formula, find the solution of the boundary value problem

$$\nabla^2 u = x^2 + y^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

subject to the boundary conditions

$$u = \frac{1}{12}(x^4 + y^4)$$

on the lines

$$x = 1; \quad y = 0; \quad y = 1 \text{ and}$$

$$12u + \frac{\partial u}{\partial x} = x^4 + y^4 + \frac{x^3}{3} \text{ on } x = 0.$$

Use central difference approximation on the boundary condition. *6*

6. (a) Find y(0·1), y(0·2) and y(0·3) from

$$\frac{dy}{dx} = x^2 - y, \quad y(0) = 1,$$

by using the 4^{th} order Taylor series method. Hence obtain y(0·4) using the Adams-Bashforth method with predictor P and corrector C as given below :

$$P : y^p_{n+1} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$C : y^c_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} - 19y'_n - 5y'_{n-1} + y'_{n-2})$$

Use one corrector iteration. *5*

(b) Show that

$$2n\, J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)].$$ 3

(c) If F^{-1} represents inverse Fourier Transform, evaluate $F^{-1}\left(\dfrac{1}{\alpha^2 + 2\alpha + 5}\right)$. 2

7\. (a) Obtain the system of equations for solving the boundary value problem

$$y'' - 5y' + 4y = 0$$

with $y(0) - y'(0) = -1, \; y(1) + y'(1) = 1$.

Use second order finite differences for y' and y'', with $h = \dfrac{1}{2}$. 5

(b) The wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ is approximated by

$$\delta_t^2\, u_i^n = r^2 \delta_x^2 \,[\theta\, u_i^{n+1} + (1 - 2\theta)\, u_i^n + \theta\, u_i^{n-1}]$$

where $r = \dfrac{k}{h}$. Using the Von Neumann method, investigate the stability. 3

(c) What type of finite elements are used for

(i) one-dimensional problems ?

(ii) two-dimensional problems ? 2