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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00385

Term-End Examination

June, 2018

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *of the remaining questions. Use of calculators is not allowed. Notations as in the study material.*

1\. (a) Prove that C_0 is a Banach space under the supremum norm. *3*

(b) Prove that the norm induced by an inner product satisfies the parallelogram law. Using this property, prove that for $p \neq 2$, l^p is not an inner product space. *3*

(c) Let X, Y be Hilbert spaces and $A \in BL(X, Y)$. Prove that R(A) is closed if and only if $R(A^*)$ is closed. *4*

2. (a) Define a reflexive space. Prove that a Banach space is reflexive if and only if its dual space is reflexive. 5

(b) State closed graph theorem. Deduce open mapping theorem from the closed graph theorem. 5

3. (a) Suppose E_1 and E_2 are closed subspaces of a normed linear space X. Is $E_1 + E_2$ closed ? Justify your answer. 3

(b) Define $T : C[0, 1] \to \mathbf{C}$ by $T(f) = f(0)$. 5

Let B[0, 1] denote the space of bounded functions on [0, 1] equipped with the supremum norm. Give a Hahn-Banach extension of T from C[0, 1] to B[0, 1].

(c) Let $X = \mathbf{R}^2$ and $A \in BL(X)$ be given by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find $\sigma(A)$. 2

4. (a) Suppose X and Y are Banach spaces and $F : X \to Y$ be a one-one bounded linear map. Prove that the range R(F) of F, is closed in Y if and only if $F^{-1} : R(F) \to X$ is bounded. 4

(b) Let $< \cdot , \cdot >$ be an inner product on a vector space X and $T \in B(X)$ be a bijection. Let 3

$$<x, y>_T := <T(x), T(y)>, x, y \in X.$$

S.T. $< \cdot , \cdot >_T$ is an inner product on X.

(c) Define $T : C[0, 1] \to C[0, 1]$ by

$$Tf(x) = \int_0^1 \sin (x - y) f(y) dy.$$

Show that the range of T is finite dimensional. Is T a compact operator ? Justify your answer. 3

5. (a) Suppose X and Y are normed spaces and $X \neq 0$. Prove that BL(X, Y) is a Banach space with the operator norm if and only if Y is a Banach space. 4

(b) Define the spectrum, eigenspectrum and the approximate eigenspectrum of a bounded linear operator $A \in BL(X)$, X is a normed linear space. Let $X = l^p$ and $A : X \to X$ be given by

$A((\alpha_1, \alpha_2, ...)) = (0, \alpha_1, \alpha_2, ...)$.

Prove that $\sigma_e(A) = \phi$. 4

(c) State the spectral theorem for a non-zero compact self-adjoint operator on a Hilbert space. 2

6. Are the following statements *True* or *False* ? Justify your answers. *5×2=10*

(a) If the dual X' of a normed linear space X is finite dimensional, then X is finite dimensional.

(b) If a vector space is complete under a norm, then it is complete under all other norms.

(c) Every bounded linear functional defined on a subspace of a Hilbert has a unique Hahn-Banach extension to the whole Hilbert space.

(d) If $\{U_\alpha\}$ is an orthonormal set in a Hilbert space H, then span $\{U_\alpha\}$ is dense in H.

(e) The eigenvalues of a positive self adjoint operator are always contained in the infinite interval $[0, \infty)$.