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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00865

Term-End Examination

June, 2018

MMT-004 : REAL ANALYSIS

Time : 2 hours | *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no. **1** is **compulsory**. Attempt any **four** questions from questions no. **2** to **6**. Calculators are **not** allowed.*

1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

   (a) The set $\left\{ \frac{m}{n} \middle| m, n \in \mathbf{N} \right\}$ is a compact subset of **R** under the usual metric.

   (b) The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$ is locally invertible at $x = 0{\cdot}25$.

   (c) The collection of all compact subsets in **R** under the usual metric is a σ-algebra of **R**.

(d) Every Lebesgue integrable function is Riemann integrable.

(e) If X and Y are metric spaces, $f : X \to Y$ is continuous and $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence.

2. (a) Suppose A and B are disjoint closed sets in a metric space (X, d). Prove that there exist disjoint open sets U and V such that $U \supset A$ and $V \supset B$. *3*

(b) Suppose E is an open set in $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$ is a function. Explain how the k^{th} derivative of f is defined, for k = 2, 3, ... Suppose $g : \mathbf{R}^n \to \mathbf{R}^m$ is linear, prove that $g^{(k)} = 0$ for k = 2, 3, ... *4*

(c) Define the Lebesgue outer measure. Prove that the Lebesgue outer measure of the empty set is zero. What is the outer measure of the set of irrationals in $\mathbf{R}$? Justify your answer. *3*

3. (a) Give, with justification, an example of a proper subspace of $\mathbf{R}^3$ which is complete under the usual metric on $\mathbf{R}^3$. On $\mathbf{R}^3$, give a metric other than the Euclidean metric and check whether $\mathbf{R}^3$ is complete under that metric. *4*

(b) State the Inverse Function Theorem for vector-valued functions. Verify the theorem for the function $f : \mathbf{R}^4 \to \mathbf{R}^4$ defined by $f(x, y, z, w) = (x + y, x^2 + y^2, wz, yw)$ at $(1, 1, 1, 1)$. *4*

(c) Consider the sequence $f_n = \chi_{[n, \infty]}$ for $n = 1, 2, 3, \ldots$ Does the Monotone Convergence Theorem hold for this sequence ? Give reasons for your answer. *2*

4. (a) Let X be a set, and d be the discrete metric on X. What are the

(i) bounded subsets of X ?

(ii) closed subsets of X ?

Give reasons for your answer.

Are the closed and bounded subsets of X, compact, if X is infinite ? Give reasons for your answer. *4*

(b) Consider the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by $f(x, y, z) = x + y + z - \sin(xyz)$. Show that $f(x, y, z) = 0$ defines a unique continuously differentiable function g on a neighborhood N of the point (0, 0) such that $f(g(u), u) = 0 \ \forall \ u \in N$. *3*

(c) Prove that every continuous function is Lebesgue measurable and that the converse is not true. *3*

5. (a) Suppose f and g are two continuous real valued functions defined on a connected metric space X. Suppose a, b ($a \leq b$), belong to the range of g and $f(x) \in [a, b]\ \forall\ x \in X$. Prove that $f(c) = g(c)$ for some $c \in X$. *4*

(b) Obtain the second Taylor's series expansion for the function $f(x, y) = x\, e^y$ at $(1, -1)$. *3*

(c) Give an example, with justification, for each of the following : *3*

(i) A stable system;

(ii) A time-varying system.

6. (a) Let (X, d) be a compact metric space and (Y, d) be any metric space. Suppose $f : X \to Y$ is continuous. Prove that f is uniformly continuous. *3*

(b) Prove that the critical points of the function $f : \mathbf{R}^2 \to \mathbf{R}$, defined by $f(x, y) = x^3 - 3xy + y^3$, are (1, 1) and (0, 0). Also prove that (0, 0) is a saddle point and (1, 1) is a local minimum. *4*

(c) Find the Fourier series of the function f defined by $f(x) = |x|$ on $[-\pi, \pi]$. *3*
