MMT-007

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

01262

June, 2017

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 50%)

Note : *Question no. 1 is **compulsory**. Attempt any **four** questions out of the remaining questions no. 2 to 7. All computations may be kept to three decimal places. Use of calculators is **not** allowed.*

1. State whether the following statements are *True* or *False*. Justify your answers with the help of a short proof or a counter-example. *5×2=10*

   (a) The radius of convergence of the series

   $$\sum_{n=0}^{\infty} \frac{x^n}{4^n}, \text{ is } \frac{1}{4}.$$

   (b) The value of $\int_{-1}^{1} (2x+1)\, P_3(x)\, dx$, where $P_3(x)$ is the third degree Legendre polynomial, is zero.

(c) The solution of the integral equation

$$F(t) = 1 + \int_a^t F(\beta) \sin (t - \beta)\, d\beta \text{ is } (t + \frac{t^2}{2}).$$

(d) Crank-Nicolson method for one-dimensional equation is stable.

(e) The order of the method

$$y_{i+1} = \frac{1}{3} [4y_i - y_{i-1}] + \frac{2h}{3} y'_{i+1}$$

for solving the initial value problem

$y' = f(x, y), y(x_0) = y_0$ is 3.

2. (a) The current I(t) and the charge Q(t) on a capacitor in an electric circuit satisfy the equations 7

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E_0, \; Q = \int_0^t I(z)\, dz .$$

The quantities L, R, C and E_0 are constants and t is the time after closing the switch in the circuit. If, initially, Q and I are zero, show, using Laplace transform technique, that

$$I(t) = \frac{E_0}{\omega L} e^{-bt} \sin \omega t$$

where $b = \dfrac{R}{2L}$ and $\omega^2 = \dfrac{1}{LC} - \dfrac{R^2}{4L^2} > 0.$

If $k^2 = \frac{R^2}{4L^2} - \frac{1}{LC} > 0$, show that

$I(t) = \frac{E_0}{kL} e^{-kt} \sinh kt.$

If $\frac{R^2}{4L^2} = \frac{1}{LC}$, show that $I(t) = \frac{R}{L} te^{-kt}$.

(b) Under usual notations, prove that 3

$$\int P_n(x)\, dx = \frac{1}{(2n+1)} \{P_{n+1}(x) - P_{n-1}(x)\} + C.$$

3. (a) Find the series solution about x = 0 of the following differential equation : 5

$(2x + x^2)\, y''(x) - y'(x) - 6x\, y(x) = 0$

(b) Using Fourier cosine transform, show that the solution to the following differential equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x < \infty, \quad t > 0$$

that is bounded and satisfies the conditions $u(x, 0) = 0,\ x \ge 0;\ \frac{\partial u}{\partial x}(0, t) = -\,a$ (constant) can be expressed as

$$u(x, t) = \frac{2}{\pi} a \int_0^{\infty} \frac{1 - e^{-ks^2 t}}{s^2} \cos sx\, ds.$$ 5

4. (a) Obtain the Green function for the boundary value problem $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = f(x)$, $y(0) = 0$, $y(1) = 0$. 5

(b) Using modified Euler method, find $y(0{\cdot}1)$, $y(0{\cdot}2)$ given that $\frac{dy}{dx} = (x^2 + y^2)$, $y(0) = 1$. 5

5. (a) Prove that

$$\int J_3(x)\,dx + J_2(x) + \frac{2}{x} J_1(x) = 0.$$ 4

(b) Find the solution of the boundary value problem

$$\nabla^2 u = x^2 + y^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

subject to the boundary conditions

$u = \frac{1}{12}(x^4 + y^4)$ on the lines

$x = 1$; $y = 0$; $y = 1$, and

$12u + \frac{\partial u}{\partial x} = x^4 + y^4 + \frac{x^3}{3}$ on $x = 0$, using the five-point formula. Assume $h = \frac{1}{2}$ along both the axes. Use central difference approximation in the boundary condition.

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6. (a) Using Picard's method of successive approximation, solve

$$\frac{dy}{dx} = (1 + xy), \;\; y(2) = 0$$

to obtain y_1 and y_2. *4*

(b) Using Milne-Simpson fourth order method with $h = 0{\cdot}1$, find $y(0{\cdot}3)$ for the initial value problem $y' = 2x + y$, $y(0) = 1$. It is given that $y(0{\cdot}1) = 1{\cdot}116$. *4*

(c) If H_n is a Hermite polynomial of degree n, then show that

$$H'_n = 4n\,(n-1)\,H_{n-2}.$$ *2*

7. (a) The heat conduction equation $u_t = u_{xx}$ is approximated by

$$\frac{1}{2k}\left(u_m^{n+1} - u_m^{n-1}\right) = \frac{1}{h^2}\left(u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}\right).$$

(i) Determine the truncation error and order of the method.

(ii) Investigate the stability using Von Neumann method. *7*

(b) Find the Fourier transform of the function $f(x) = e^{-4x^2}$. *3*
