M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00062

Term-End Examination

June, 2017

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. 1 is **compulsory**. Attempt any **four** questions from questions no. 2 to 7. Calculators are **not** allowed. Notations as in the study materials.*

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1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) If $A = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$, then under the standard metric on $\mathbf{R}$, $d(0, A) = 1$.

(b) For the collection $\{B_{m, n} : m, n \in \mathbf{Z}\}$,

$B_{m, n} = \{ (x_1, x_2) \in \mathbf{R}^2 \mid$

$\sqrt{(x_1 - m)^2 + (x_2 - n)^2} < \frac{1}{2}\}$ covers $\mathbf{R}^2$.

(c) The only component in a discrete metric space is the whole set.

(d) If the partial derivatives of a function $f : \mathbf{R}^2 \to \mathbf{R}$ exist at (x_0, y_0), then f is differentiable at (x_0, y_0).

(e) The measure of the set of irrationals is zero.

2. (a) State Urysohn's lemma. Using the lemma, deduce that for any two disjoint closed sets A, B in a metric space (X, d), there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$. *4*

(b) Let E be an open subset of $\mathbf{R}^n$. For a map $f : E \to \mathbf{R}^m$, define the partial derivatives of f. Find the partial derivatives of the function $f : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $f(x, y, z) = (xy, z)$. *3*

(c) Define the Lebesgue outer measure of a set $A \subset \mathbf{R}$. Also, find the outer measure of the following sets : *3*

(i) $[\frac{1}{2}, 5] \cup [\frac{1}{2}, 20] \cup \{22\}$

(ii) $\{x \in \mathbf{Z} \mid x \text{ is even}\}$

3. (a) Prove that a closed subset of a complete metric space is complete. Is a complete subset of a metric space closed ? Justify your answer. *4*

(b) Check whether the conditions of the inverse function theorem are satisfied for the following function :

$f : \mathbf{R}^3 \to \mathbf{R}^3$ at the point (0, 1, 1) :

$$f(x, y, z) = (e^x \cos z, e^x \sin z, x + y + z)$$

Is f locally invertible at (0, 1, 1) ? Justify your answer. *3*

(c) Let $f : [0, 1) \to [0, \infty]$ be a measurable function. Define $f_n : [0, 1) \to [0, \infty]$ by

$$f_n(x) = \begin{cases} f_n(x) - \frac{1}{n}, & \text{if } f(x) \geq \frac{1}{n} \\ 0, & \text{if } f(x) < n \end{cases}$$

Show that $\int_{-\infty}^{\infty} f_n \to \int_{-\infty}^{\infty} f$ as $n \to \infty$. *3*

4. (a) Show that the continuous image of a connected metric space is connected. Hence deduce that the set $\{(x, y) : \frac{x^2}{4} + \frac{y^2}{9} = 1\}$ is a connected subset of $\mathbf{R}^2$. *3*

(b) Obtain the second Taylor series expansion of the function $f(x, y) = x^2y + 3xe^y$ at (1, 0). *4*

(c) Find the Fourier series of the function f defined by

$$f(x) = \begin{cases} -x^2, & -\pi < x \leq 0 \\ x^2, & 0 < x < \pi \end{cases}.$$ *3*

5. (a) Prove that a countable intersection of dense open sets in a complete metric space is non-empty. *4*

(b) Find the critical points of the function $f(x, y, z) = x^3 + y^3 - 3x - 12y + 20$ and check whether they are extreme points. *4*

(c) Show that $L^2(E) \subseteq L^1(E)$ where E is any set having finite measure. *2*

6. (a) Check whether the set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is dense, or nowhere dense in **R** under the standard metric. *3*

(b) State the implicit function theorem for a function $f : E \to \mathbf{R}$ where E is an open subset of $\mathbf{R}^2$. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$f(x, y) = y^3 - yx^2 - 2x^5$. Show that there exists a differentiable function g in a neighbourhood of 1 such that $g(1) = -1$. 4

(c) Prove that for a measurable function f,

$$\left| \int f\,dm \right| \leq \int |f|\,dm.$$ 3

7. (a) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence in **R**. Does it converge in **R**. ? Justify your answer. 3

(b) Check whether the system $(Rf)(t) = a + f(t)$ is linear or not. 2

(c) Define signals and systems. Explain translation systems, scaling systems and reflection systems using suitable examples. 5
