No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00346

Term-End Examination

June, 2016

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is **not** allowed.*

1. (a) Consider a closed queuing system with 3 stations and 2 clients. Stations are in cascade and from the last queue, clients recycle to the first one. All queues have exponential services have no queuing line, thus, when a client arrives to one of the first two stations and finds the server occupied, he proceeds immediately to the following queue.

   (i) Find the steady state distribution, if it exists.

   (ii) Find the average number of clients in each of the queues.

   (iii) Find the average time spent by clients in each of the queues. 7

(b) Let $\{X_n\}_{n \in N_0}$ be a Markov chain with the transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(i) Find all stationary distributions.

(ii) If the chain starts from the state $i = 1$, what is the expected number of steps before it returns to 1 ?

(iii) How many times, on an average, does the chain visit 2 between two consecutive visits to 1 ? *8*

2. (a) Let $\{Z_n\}_{n \in N_0}$ be a Branching process with state space $S = \{0, 1, 2, 3, 4, ...\} = N_0$, and the probability of each offspring is 2. Classify the states and describe all closed sets. *5*

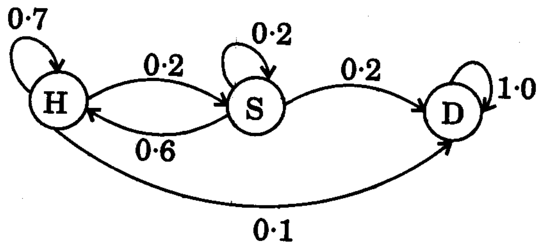
(b) Consider the orthogonal transformation of the correlated zero mean random variables x_1 and x_2 and consider

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If $E(x_1^2) = \sigma_1^2$, $E(x_2^2) = \sigma_2^2$ and

$E(x_1 x_2) = \rho\sigma_1\sigma_2$, determine the angle θ such that y_1 and y_2 are uncorrelated. *4*

(c) The transition graph of a Markov chain having three states, healthy, sick and dead, which provides transition probabilities for the changes in a week in the condition of a patient, is given below :



If the probability that a patient is healthy be 0·8, then

(i) find the probability that he/she will be sick in the coming first week.

(ii) find the probability that he/she will remain healthy in the next two weeks.

Also write the transition probability matrix. *6*

3. (a) State the assumptions of the Poisson process. Taxis arrive at a spot from north at a rate of 40 per hour and from south at a rate of 60 per hour in accordance with independent Poisson process. Find the probability that a person will have to wait for a taxi at the spot more than 2 minutes. How many taxis will arrive at the spot in 10 minutes on an average ? *6*

(b) Joint probability distribution of two random variables X_1 and X_2 are given in the following table :

$X_1 \backslash X_2$	0	1
–1	0·16	0·14
0	0·04	0·26
1	0·10	0·30

Find

(i) mean vector,

(ii) variance-covariance matrix,

(iii) correlation matrix. *9*

4. (a) A Markov chain has the following transition matrix :

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Determine the probabilities of ultimate return to the states and mean recurrence times of the states. Check whether the chain is irreducible. *6*

(b) A random sample of size 4 from $N_2(\mu, \Sigma)$ is given below :

x_1	2	8	6	8
x_2	12	9	9	10

Test the hypothesis $H_0 : \mu' = [7, 11]$ against $H_1 : \mu' \neq [7, 11]$ at 5% level of significance.

[You may like to use the values

$F_{2, 4}(0{\cdot}05) = 19{\cdot}25$ and $F_{2, 2}(0{\cdot}05) = 19{\cdot}00$] *9*

5. (a) Variables x_1, x_2 and x_3 have the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}63 & 0{\cdot}4 \\ 0{\cdot}63 & 1 & 0{\cdot}35 \\ 0{\cdot}4 & 0{\cdot}35 & 1 \end{bmatrix}$$

Write its factor model. *9*

(b) Suppose in a branching process, the offspring distribution is given as

$$p_k = {}^nC_k\, p^k\, q^{n-k};\ 0 < p < 1,\ \ q = 1 - p,$$
$$k = 0, 1, 2, \ldots$$

What will be the probability of extinction of this branching process ? *3*

(c) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. The number of people in each family is independent and takes the values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and variance of the number of individuals migrating to this area during a fixed five-week period. *3*

6. (a) Let $[X_1, X_2, X_3]$ be distributed as $N_3(\mu, \Sigma)$,

$$\text{where } \mu = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Find

(i) the conditional densities $f(x_2/x_1, x_3)$ and $f(x_1, x_2/x_3)$,

(ii) the distribution of $z = 2x_1 + x_2 - x_3$. *8*

(b) Calculate the least square estimate b, the residual e and the residual sum of squares for a straight-line model $Y = b_0 + b_1X + e$ fitted to the following data : *7*

X	0	1	2	3	4
Y	1	4	3	8	9

7. (a) Let X_1, X_2 and X_3 be random variables with means 3·3, – 3·1 and 2·5 respectively and the variances 9, 16 and 25 respectively. Also let $\rho_{12} = 0{\cdot}5$, $\rho_{13} = 0{\cdot}3$, $\rho_{23} = -0{\cdot}4$, where ρ_{ij} is the correlation coefficient between x_i and x_j.

(i) Write down the mean vector and the variance-covariance matrix of $X = [X_1 \;\; X_2 \;\; X_3]^t$.

(ii) Find the mean and variance of $1^t x$,

$$\text{where } 1 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(iii) Also, find the covariance between $1^t x$

$$\text{and } m^t x, \text{ where } m = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, 1 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ *10*

(b) Let $X = [X_1 \;\; X_2 \;\; X_3]$ having the covariance

$$\text{matrix } \Sigma = \begin{bmatrix} 4 & 4 & -2 \\ 4 & 9 & 1 \\ -2 & 1 & 16 \end{bmatrix}.$$

Find the correlation matrix. *5*

8. State whether the following statements are *true* or *false*. Give short proof or counter-example to support your answer. 10

(a) In a Markov chain $\{X_n\}_{n \in N_0}$ with state space S, if all rows of transition matrix are equal, then all states belong to the same class.

(b) If (X, Y, Z) be a trivariate random variable, where X, Y and Z are independent uniform random variables over (0, 1), then $P(Z \geq XY) = \frac{3}{4}$.

(c) If the joint pdf of a bivariate random variable (X, Y) is

$$f_{XY}(x, y) = \frac{1}{2\sqrt{3}\pi} \exp\left[-\frac{1}{2}(x^2 - xy + y^2 + x - 2y + 1)\right]$$

$$-\infty < x, y < \infty,$$

then the var(X, Y) = 1 and coefficient of correlation is $\frac{1}{2}$.

(d) If A is a 3×3 transition probability matrix, then the sum of the entries of the matrix A^3 is 3.

(e) A finite Markov chain has a null persistent state.