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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

00826

M.Sc. (MACS)

Term-End Examination

June, 2014

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage 70%)

Note : *Question 1 is* ***compulsory****. Attempt any* ***four*** *of the remaining questions. Use of calculator is* ***not*** *allowed.*

1. State whether the following statements are *true* or *false*. Give a brief justification with a short proof or a counter example. *2×5=10*

(a) The closure of a linear subspace in a normed linear space is again a linear subspace.

(b) The map $P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$P(x_1, x_2, x_3) = (x_1, x_2, 0)$, is an open map.

(c) l^1 is a reflexive space.

(d) Every Hilbert space is strictly convex.

(e) Every linear isometry on a Hilbert space is a unitary operator.

2. (a) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a linear space X such that for some $r > 0$ $\|x\|_2 \leq r\|x\|_1$ for all $x \in X$. Then show that $B_1(0, 1) \subset B_2(0, r)$ where $B_1(0, 1)$ and $B_2(0, r)$ respectively denote the open balls of radius 1 and r with centre at O. Deduce the condition for the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$. *4*

(b) Let S be a subset of a Hilbert space H. Show that $S^{\perp} = S^{\perp\perp\perp}$. *3*

(c) Define $A : \mathbb{C}^3 \to \mathbb{C}^3$ by $A(z_1, z_2, z_3) = (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3)$, $\alpha_j \in \mathbb{C}$. When is A a positive operator ? *3*

3. (a) Let X be a normed linear space and M be a closed linear subspace of X. Let $x_0 \in X$ be such that $x_0 \notin M$ and $r = d(x_0, M)$. Show that there is a bounded linear functional f on X such that $f(x_0) = 1$, $f(y) = 0$ for all $y \in M$ and $\|f\| = \frac{1}{r}$. *4*

(b) Let X be a normed space and Y be a Banach space.

(i) Define the operator norm on the linear space B(X, Y).

(ii) Show that B(X, Y) is complete under this norm.

(iii) Use the result in (ii) to show that the dual space of every normed space is a Banach space. *6*

4. (a) Let X be a Banach space and Y be a normed space and $\mathscr{A}$ be a subset of BL(X, Y) such that for every $x \in X$, there exists a positive real number k_x such that $\| F(x) \| \leq k_x \ \forall \ F \in \mathscr{A}$. Then show that $\sup \{\| F \| : F \in \mathscr{A}\} < \infty$. *4*

(b) Let M, N be closed linear subspaces of a Hilbert space H. If $M \perp N$, prove that $M + N$ is closed. *3*

(c) Let $A : l_2 \to l_2$ be defined by $A(x_1, x_2, ...) = (0, 0, x_3, x_4, ...)$. Prove that A is self-adjoint, positive. Also find $\sqrt{A}$. *3*

5. (a) Prove that the dual of $(\mathbb{R}^2, \ \| \cdot \|_1)$ is isometrically isomorphic to $(\mathbb{R}^2, \ \| \cdot \|_\infty)$. *5*

(b) If $\{e_n\}$ is an orthonormal sequence in a Hilbert space H and if A is a compact operator on H, show that $Ae_n \to 0$. *3*

(c) Let X = C[0, 1]. Find a bounded linear operator T on X whose spectrum $\sigma(T)$ is [2, 3]. *2*

6. (a) Let $a_1, a_2 \in \mathbb{R}$. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $\| x \| = | a_1 x_1 | + | a_2 x_2 |$. Is this always a norm on $\mathbb{R}^2$? Justify your answer. *3*

(b) (i) State Riesz Representation Theorem for Hilbert spaces.

(ii) Let $H = \mathbb{R}^3$ and let $f : H \to \mathbb{R}$ be given by $f(x_1, x_2, x_3) = x_1$. Find a $y \in H$ that represents f. *3*

(c) Let X and Y be two normed spaces and $T : X \to Y$ be a linear, continuous and surjective operator. Prove that the operator $\tilde{T} : X/\text{Ker } T \to Y$,

$\tilde{T}(x + \text{Ker } T) = Tx \ \forall \ x \in X$, is a well-defined, bijective, linear and continuous operator. *4*

7. (a) Define $T : C[0, 1] \to C[0, 1]$ by

$$Tx(t) = tx(t), \ 0 \le t \le 1.$$

Prove that T is a bounded linear map and compute its norm. *3*

(b) State Projection Theorem for Hilbert spaces. Illustrate with an example. *3*

(c) Let A be a bounded linear operator on a Hilbert space H. Obtain the relations between Z(A) and R(A*) and between R(A) and Z(A*). *4*