MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00395

June, 2014

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Weightage : 70%

Note : *Question no. 1 is* ***compulsory****. Do any* ***four*** *questions out of question nos. 2 to 7.*

1. State, whether the following statements are True or False. Give reasons for your answers. *5×2=10*

(a) If (X, d_1) is a discrete metric space and (Y, d_2) is any metric space, then any function $f : X \to Y$ is continuous.

(b) Continuous image of a Cauchy sequence in a metric space is a Cauchy sequence.

(c) Discrete metric space has no dense subset.

(d) Outer measure of a subset of **R** is always finite.

(e) If $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} |x| + |y|, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{elsewhere} \end{cases}$$

then the directional derivative of f along (0, 1) does not exist at (0, 0).

2. (a) Let $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be defined by

$$d(x, y) = \max_{1 \le i \le n} |x_i - y_i|, \quad x, y \in \mathbf{R}^n.$$

Show that d is a metric on $\mathbf{R}^n$. *4*

(b) Verify whether the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$f(x, y, z) = \begin{cases} (y, x, z^2 \sin \frac{1}{z}) & \text{if } (x, y, z) \neq (0, 0, 0) \\ (0, 0, 0) & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

is continuously differentiable or not. *3*

(c) Find the outer measure of the following sets : *3*

(i) $A = \{x \in \mathbf{R} \mid \cos 2x = 1\}$

(ii) $B = \{x \in \mathbf{R} \mid |x - 5| \le 7\}$

3. (a) State Urysohn's Lemma. Use this lemma to prove the following result :

Let E and F be non-empty disjoint closed subsets of a metric space (X, d). Then, there exist open sets $A \supset E$ and $B \supset F$ such that $A \cap B = \phi$. *5*

(b) Verify the Implicit Function Theorem for the function $f : \mathbf{R}^5 \to \mathbf{R}^2$ defined by

$$f(x_1, x_2, y_1, y_2, y_3) = (2e^{x_1} + x_2y_1 - 4y_2 + 3,\ x_2 \cos x_1 - 6x_1 + 2y_1 - y_3)$$

at (0, 1, 3, 2, 7). *5*

4\. (a) Prove that a Cauchy sequence in a metric space is convergent if and only if it has a convergent subsequence. *4*

(b) Find the critical points of the function, f, given by

$$f(x, y, z) = 2x^2 - 2y^2 + 4yz - 3z^2 - x^4 + 5$$

and check whether they are extreme points. *4*

(c) Show that a finite set in a metric space is nowhere dense. *2*

5\. (a) Use Lagrange Multipliers Method to find the point on the plane $2x - 2y + z = 4$ that is closest to the origin. *4*

(b) Find the components of $\boldsymbol{\theta}$ under the usual metric. *2*

(c) State Monotone Convergence Theorem. Verify the theorem for $\{f_n\}$ where $f_n : \mathbf{R} \to \mathbf{R}$ is defined by

$$f_n(x) = \begin{cases} 1 & , \ x \in \left[\dfrac{1}{n+1},\ 1 - \dfrac{1}{n+1}\right] \\ 0 & , \ \text{elsewhere} \end{cases}$$ *4*

6. (a) Obtain the Taylor's series expansion for $f(x, y) = e^{xy}$ at (1, 1). *3*

(b) Define the following in the context of signals and systems and give one example for each : *4*

(i) Stable system

(ii) Time-invariant system

(c) Find the interior, closure and boundary of the set $A = \{(x, 0) \in \mathbf{R}^2 : 0 < x \le 1\}$ as a subset of $\mathbf{R}^2$ with metric standard. *3*

7. (a) Show that every compact subset of a metric space is complete. *3*

(b) Find the Fourier series for *5*

$$f(x) = \begin{cases} -\pi & \text{for } -\pi < x < 0 \\ x & \text{for } 0 < x < \pi \end{cases}$$

(c) Prove that if $g(x) = f(-x)$, then $\hat{g}(\omega) = \hat{f}(\omega)$. *2*