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M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

June, 2012

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

Note : *Answer question number 1 which is* ***compulsory****. Attempt* ***any four*** *of the remaining six questions.*

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1. Are the following statements ***true or false*** ? Justify your answer with the help of a short proof or a counter example. **5x2=10**

   (a) The closed unit ball in l^2 is compact.

   (b) Any complete subspace of a normed linear space is closed.

   (c) A finite dimensional normed linear space is necessarily reflexive.

   (d) l' is not a Hilbert space.

   (e) A normal operator is self adjoint.

2. (a) Suppose Y is a closed subspace of a normed linear space X and $x_0 \notin Y$, $x_0 \in X$. Prove that there is a linear functional $f_0 \in X^1$ such that $f_0(x_0) = d(x_0, y)$, $f_0(Y) = 0$ and $\|f_0\| = 1$. **4**

(b) Suppose $\|.\|_1$ and $\|.\|_2$ are two equivalent norms on a vector space X. If $(X, \|.\|_1)$ is complete, then show that so is $(X, \|.\|_2)$. 3

(c) Define f on C[0,1] by : 3

$f(x) = \int_0^{1/2} x(t)dt,\ x \in C([0,1]$. Prove that f is a bounded linear functional on C[0,1].

3. (a) Define $T : l^2 \to l^2$ by $Tx(n) = \frac{1}{n} x(n)$, n = 1,2,... 3

Show that T is compact and positive with eigenvalues $\frac{1}{n}$, n = 1, 2, ...

(b) Let $X = l'$ and $A \in BL(x)$ be defined by $A(x) = (0, x(1), x(2) \ldots\ldots\ldots)$. Find σ_i (A). 2

(c) State and prove the Riesz Representation Theorem for Hilbert spaces. 5

4. (a) Give a bounded linear non zero map that is not an open map. Does this contradict the open Mapping Theorem ? Justify your answer. 4

(b) If $\{e_n\}$ is an orthonormal basis for a Halbert space H, prove that $\| x \|^2 = \Sigma \mid \langle x, e_n \rangle \mid^2$ for all x in H. 3

(c) Consider $\mathbf{R}^2$ and $\mathbf{R}^3$ with the Euclidean norms. Compute $\| T \|$ for $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by 3

$T(x_1, x_2) = (x_1, x_2, x_1 + x_2), x_i \in \mathbf{R}.$

5. (a) Prove that the dual space of a reflexive space is also reflexive. 4

(b) Show that a bounded linear operator A on complex Hilbert space H is self adjoint if and only if $< Ax, x >$ is real for all x in H. 3

(c) If $x \perp y$, then show that $\|x+y\| = \|x-y\|$. 3

6. (a) Let $A \in BL(H)$ be a compact operator, then show that A^*A is a compact 3

(b) Let X be a Banach space and let $\{T_n\}$ be a sequence of bounded linear operators on X such that $Tx = \lim T_n x$ exists for all x in X. Show that T is a bounded linear operator on X. 3

(c) Give an example each : a self adjoint operator $\neq 0$, I and a unitary operator. 4

7. (a) Let X be a normed linear space such that its dual X' is separable, then show that X is separable ? Is the converse true ? Justify your answer. 5

(b) Let E_1 be a closed subspace and E_2 be a finite dimensional subspace of a normed subspace X. Prove that $E_1 + E_2$ is closed in X. 3

(c) Give an example of an unbounded linear operator. Justify your choice of example. 2
