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MMT-006

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M. Sc. (MACS)

Term-End Examination

December, 2021

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question No.* ***6*** *is compulsory.*

(ii) Attempt any ***four*** *of the remaining questions.*

(iii) Calculators are not allowed.

(iv) Notations as in the study material.

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1. (a) Find an orthonormal basis for $\mathbf{R}^3$ with inner product : 3

$$< x, y > = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 .$$

(b) Show that C [0, 1] with norm $\|f\| = |f(0)| + \|f\|_\infty$ is a Banach space. 3

(c) State the closed graph theorem. If A is a linear map on a Hilbert space H such that $< Ax, y > = < x, Ay >$ for all x, y in H, prove that A is continuous. 4

2. (a) Find a bounded linear functional f on l^3 such that $f(e_3) = 3$ and $\|f\| = 3$. 3

(b) Show that a bounded linear operator on a Hilbert space has a nontrivial closed invariant subspace if and only if its adjoint has one such subspace. 3

(c) State the Riesz representation theorem for Hilbert spaces. Deduce that a family $\{f_i\}$ of bounded linear functionals on a Hilbert space H is bounded if the set $\{f_i(x)\}$ of scalars is bounded for every x in H. 4

3. (a) Prove that $l^1 \subset l^2$. If :

$$\mathrm{T} : (l^2, \|.\|_2) \to (l^1, \|.\|_2)$$

is a compact operator, show that :

$$\mathrm{T} : (l^2, \|.\|_2) \to (l^2, \|.\|_2)$$

is also compact. 4

(b) Prove that there is a closed linear subspace N of c such that $c = c_b \oplus \mathrm{N}$. 3

(c) Let D be a dense linear subspace of a normed linear space X. Without using Hahn-Banach theorem, show that every bounded linear functional f on D has a bounded linear extension g on X with $\|g\| = \|f\|$. 3

4. (a) Consider $(\mathbf{R}^2, \|.\|_1)$ and let M = $\{(x, y) \in \mathbf{R}^2 : x + y = 0\}$. For $p = (1, 0)$, find two distinct points q_1, q_2 such that $d(p, M) = d(p, q_1) = d(p, q_2)$. What happens if we take $\|.\|_2$ in place of $\|.\|_1$? 5

(b) Prove that a bounded linear operator A on a normed linear space has rank 1 (i.e. dim R (A) = 1) if and only if there are $0 \neq x_0 \in X$ and a nonzero bounded linear functional f on X such that $Ax = f(x)\, x_0$ for all x in X. 5

5. (a) Let A be a bounded linear operator on a Hilbert space H. If A is bijective, show that R (A*) is dense and A* is bounded below and hence conclude that A has a bounded inverse. 6

(b) Determine the dual space of $\left(\mathbf{R}^2, \|.\|_\infty\right)$. 4

6. State with justification, whether the following statements are True *or* False :

(a) If M is a linear subspace of a Hilbert space H such that $M^\perp = (0)$, then M = H. 2

(b) If φ is a bounded linear functional on a normed linear space X, then there is a closed linear subspace N of X such that $X = \ker \varphi \oplus N$. 2

(c) There is a bounded linear operator $A \neq 0$ on l^1 such that $\sigma(A) = \{0\}$. 2

(d) If a Banach space X has a closed, reflexive subspace, then X is reflexive. 2

(e) If A and B are positive operators on a Hilbert space H, then AB is a positive operator on H. 2

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