M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

February, 2021

MMT-003 : ALGEBRA

Time : 2 hours *Maximum Marks : 50*

Note : *Question no.* ***1*** *is* ***compulsory****. Attempt any* ***four*** *questions from Questions no. 2 to 6. Use of calculators is* ***not*** *allowed.*

1. State whether the following statements are *TRUE* or *FALSE*. Give reasons for your answers. *10*

(i) There exists an extension field of $\mathbf{Z}_2$ of order 25.

(ii) Every group of order 18 has a normal subgroup of order 2.

(iii) $9 = 1 + 1 + 1 + 3 + 3$ is the class equation of some group of order 9.

(iv) Every symplectic matrix over $\mathbf{R}$ of order 2 is an orthogonal matrix.

(v) $\mathbf{R}[x, y]$ is a PID.

2. (a) Check whether or not $\mathbf{Q}(\omega) = \mathbf{Q}(\omega^2)$, where ω is a non-real cube root of unity. Also find $[\mathbf{Q}(\omega) : \mathbf{Q}]$. *3*

(b) Let G be a group of order 39. Suppose that G acts on a set X having 14 elements. What are the possible values of $|O_x|$ for $x \in X$? Show that there exists an $x_0 \in X$ such that $O_{x_0} = \{x_0\}$. *4*

(c) Check whether or not

$\rho : A_n \to \mathbf{R} \setminus \{0\} : \rho(\sigma) = \sigma(n)$ is a

representation of A_n, for $n \in \mathbf{N}$. *3*

3. (a) Given that G is a simple group of order 168, find the number of Sylow 7-subgroups in G. How many elements are there in G having order equal to 7 ? Give reasons for your answer. *4*

(b) Check whether 978-93-82050-72-4 is a valid ISBN number. *3*

(c) Check whether or not $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ is a Galois extension of $\mathbf{Q}$. Is it a Galois extension of $\mathbf{Q}(\sqrt{3})$? Give reasons for your answer. *3*

4. (a) Find all positive integers x in [1, 200] such that $x \equiv 2 \pmod 3$, $x \equiv 1 \pmod 5$, $x \equiv 3 \pmod 7$. *4*

(b) Decompose $\mathbf{M}_2(\mathbf{C})$ into orbits under the action of left multiplication by $GL_2(\mathbf{C})$. Find the stabiliser of $\begin{bmatrix} 1 & i \\ 0 & 2 \end{bmatrix}$ under this action. *6*

5. (a) Find the elementary divisors and invariant factors of the group $\mathbf{Z}_6 \times \mathbf{Z}_{15} \times \mathbf{Z}_{21}$. *4*

(b) Check whether or not $\mathbf{Z}[\sqrt{-3}]$ is a UFD. *4*

(c) How many Sylow 3-subgroups does an abelian group of order 210 have ? Give reasons for your answer. *2*

6. (a) Prove that if G is a group, then there is a free group F such that $G \simeq \frac{F}{H}$, for some $H \Delta F$. *4*

(b) Find $[K : \mathbf{Q}]$, where $K = \mathbf{Q}(5^{1/3}, 11^{1/5})$ giving detailed reasons for your answer. Also check whether or not $x^5 - 11$ is irreducible over $\mathbf{Q}(5^{1/3})$. *6*
