No. of Printed Pages : 7 **MMT-007**

M. Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M. Sc. (MACS)

Term-End Examination

December, 2020

MMT-007 : DIFFERENTIAL EQUATIONS AND NUMERICAL SOLUTIONS

Time : 2 Hours *Maximum Marks : 50*

Note : *(i) Question* ***No. 1*** *is* ***compulsory****.*

(ii) Answer any ***four*** *questions out of remaining Q. Nos.* ***2*** *to* ***7****.*

(iii) Use of scientific and non-programmable calculator is allowed.

1. State whether the following statements are true *or* false. Justify your answer with the help of a short proof or a counter example. No marks will be awarded without justification : 2×5=10

(a) Initial value problem :

$$\frac{dy}{dx} = f(x, y), \ y(x_0) = y_0$$

is equivalent to the integral equation $y(x) = \int_{x_0}^{x} f(t, y(t))\, dt$.

(b) If L denotes Laplace transform, and if :

$$L\{f(t)\} = f(s)$$

and $G(t) = \begin{Bmatrix} F(t-a) & t > a \\ 0 & t < a \end{Bmatrix}$

then L G $(t) = e^{-as} f(s)$.

(c) The partial differential equation :

$$\frac{\partial^2 u}{\partial x^2} + 4x \frac{\partial^2 u}{\partial y\, \partial x} + (1 - y^2) \frac{\partial^2 u}{\partial y^2} = 0$$

is parabolic inside the ellipse $4x^2 + y^2 = 1$.

(d) The interval of absolute stability of the Runge-Kutta method :

$$y_{i+1} = y_i + (-k_1 + 2k_2)$$

$$k_1 = hf(x_i, y_i), k_2 = hf\left(x_i + \frac{h}{4}, y_i + \frac{1}{4} k_1\right)$$

is $-2 < \lambda h < 2$.

(e) For Legendre polynomials :

$$\sum_{h=0}^{\infty} P_n(-x)\, h^n = (1 - 2xh + h^2)^{-\frac{1}{2}}$$

2. (a) Find the series solution about $x = 0$ of the differential equation : 6

$$ax(1+x)y'' - 6y' + 2y = 0$$

(b) If :

$$f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ x & \text{for } 0 < x < 1 \end{cases},$$

show that :

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \ldots\ldots,$$

where $P_n(x)$ is a Legendre polynomial of degree n. 4

3. (a) Using the method of variation of parameters, determine the appropriate Green's function for the boundary value problem $y'' + y + f(x) = 0, y'(0) = 0$ $y(2) = 0$, and express the solution as a definite integral. 6

(b) Find the appropriate value of $y(0.4)$ for initial-value problem $y' = x - y^2, y(0) = 1$. using multi-step method $y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$ with step length $h = 0.2$. Compute the starting value using Taylor's series method (second order) with same step-length. 4

4. (a) Using Laplace transform, solve the p.d.e. :

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$$

subject to the conditions : 6

$$u(0,t) = 10 \sin 2t,$$

$$u(x,0) = 0,$$

$$u_x(x,0) = 0 \lim_{x \to \infty} u(x,t) = 0$$

(b) If $f'(x_k)$ is approximated by :

$$f'(x_k) = a f(x_{k+1}) + bf'(x_{k+1}),$$

find the values of a and b. What is the order of approximation ? 4

5. (a) Using the Crank Nicolson method, integrate upto one time level for the solution of the initial value problem : 5

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1$$

with $u(x, 0) = \sin 2\pi x,$

$$u(0, t) = 0 = u(1, t)$$

with $h = \frac{1}{3}$ and $\lambda = \frac{1}{6}$.

(b) Using the substitution $z = \sqrt{x}$, reduce the equation : 3

$$xy'' + y' + \frac{y}{4} = 0$$

to Bessel's equation. Hence find its solution.

(c) If H_n is a Hermite polynomial of degree n, then show that : 2

$$H_n'' = 4n(n-1)H_{n-2}$$

6. (a) Solve the initial-value problem :

$$y' = -2xy^2 \text{ and } y(0) = 1$$

with $h = 0.2$ on the interval $\left[0, 0.4\right]$ using the predictor-corrector method :

$$\text{P} : y_{i+1} = y_i + \frac{h}{2}\left(3y_i' - y_{i-1}'\right)$$

$$\text{C} : y_{i+1} = y_i + \frac{h}{2}\left(y_{i+1}' - y_i'\right)$$

Perform two corrector iterations per step. Use the exact solution $y\left(x\right) = \left(1 + x^2\right)^{-1}$ to obtain the starting value. 4

(b) Using a second order finite difference method, solve the boundary value problem : 4

$$x^2 y'' = 2y - x$$

$$y\left(2\right) = 0$$

$$y\left(3\right) = 0$$

with $h = \frac{1}{3}$.

(c) Find : 2

$$\text{L}^{-1}\left\{\cot^{-1} z\right\}$$

7. (a) Using Laplace transform, solve the following initial value problem : 5

$$y_1'' = y_1 + 3y_2; \; y_2'' = 4y_1 - 4e^t$$

with $y(0) = 2, y_1'(0) = 3,$

$$y_2(0) = 1,$$

$$y_2'(0) = 2.$$

(b) Consider a steel plate of size 15 cm × 15 cm. If two of its parallel sides are held at 100°C and the other two parallel sides are held at 0°C, determine the steady state temperature at interior points assuming a grid size of 5 cm × 5 cm. Use fire-point formula.

5