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MMT-006

# M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

## M.Sc. (MACS)

## Term-End Examination, 2019

## MMT-006 : FUNCTIONAL ANALYSIS

**Time : 2 Hours** **Maximum Marks : 50**

**Weightage : 70%**

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**Note : Question No. 6 is compulsory.** Attempt **any four** of the remaining questions.

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1. (a) Find an inner product on $\mathbb{R}^3$ such that the unit sphere is the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, a, b, c > 0.$$ [3]

(b) If the unit sphere $\{x \in X : \|x\| = 1\}$ in a normed space $X$ is complete, prove that $X$ is complete. [4]

(c) Use F.Riesz Lemma to show that the closed unit ball in an infinite dimensional Banach space is not compact. [3]

2. (a) $H$ is a Hilbert space, $A : H \to H$ is linear and $< Ax, y > = < x, Ay >$ for all $x, y \in H$. Use the closed graph theorem to show that $A$ is continuous. [3]

(b) Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent on $C[0,1]$. [4]

(c) Prove directly that the dual space of a normed space is complete. [3]

3. (a) Let $\{e_n\}$ be the canonical basis in $l^2$ and let $u_n = \frac{n}{n+1} e_n$ for all $n$. Show that every $x \in l^2$ has a norm convergent expansion of the form $x = \sum a_n u_n$. [3]

(b) Let $f$ be a bounded linear functional on a subspace $M$ of a normed space $X$ and let $x_0 \in X, x_0 \notin M$. Prove that $f$ has an extension $g$ to $N =$ span $M \cup \{x_0\}$ such that $\|g\| = \|f\|$. [3]

(c) Show that a linear map $T : X \to Y$ between normed spaces is continuous if and only if $\{Tx_n\}$ is a Cauchy sequence in $Y$ for every Cauchy sequence $\{x_n\}$ in $X$. [4]

4. (a) If $M$ is a closed subspace of a Hilbert space $H$, then $H = M \oplus M^{\perp}$. Prove. [4]

(b) $\{T_n\}$ is a sequence of bounded linear maps on a Banach space $X$ and $Tx = \lim_{n \to \infty} T_n x$ exists for all $x \in X$. Show that $T$ is a bounded linear operator. [3]

(c) Calculate $\|T\|$ where $T$ is defined on $\left(C[0,1], \|\cdot\|_{\infty}\right)$ by $Tf(x) = \int_0^x f(t)dt,\ 0 \le x \le 1$. [3]

5. (a) Let $H$ be a Hilbert space and fix $u, v \in H$. Define $Ax = < x, u > v,\ x \in H$. Prove that $A$ is compact and determine $A^*$. [4]

(b) Let $X$ be a reflexive Banach space show that $X'$ is reflexive. [2]

(c) Let $M = \{(x_n) \in l^2 : x_1 + x_2 = 0\}$. Show that $M$ is a closed subspace and find an orthonormal basis for $M$. [4]

6. State whether the following statements are True or False. Justify your answers : [5x2=10]

(a) The sequence $\{u_n = e_1 + ....... + e_n\}$ has no limit in $l^1$.

(b) If $f$ is a linear functional on $\mathbb{R}^2$ such that $f(1,1) = 0 = f(1, -1)$, then $f \equiv 0$.

(c) It follows from the open mapping theorem that any non-zero linear functional on a Banach space is an open map.

(d) If $M$ is a closed subspace of a Hilbert space $H$, then the projection theorem implies $\frac{H}{M}$ is a Hilbert space.

(e) For a bounded linear map $A$ on a Banach space, if $\sigma(A) = \{0\}$, then $A = 0$.

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