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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

00942

Term-End Examination

December, 2018

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is* ***compulsory****. Answer any* ***six*** *questions from questions no. 1 to 7. Use of calculator is* ***not*** *allowed. All the symbols used have their usual meaning.*

1. (a) The joint probability mass function of two random variables X and Y is given in the following table :

X \ Y	−1	0	1
1	0·2	0·2	0·2
2	0·2	0·1	0·1

(i) Find E(X), E(Y), V(X) and V(Y).

(ii) Test independence of X and Y.

(iii) Find $E(X \mid Y = 0)$ and $V(Y \mid X = 1)$.

(iv) Obtain Cov (X, Y). *10*

(b) Suppose that families migrate to an area at a Poisson rate of $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$, then find the expected value and variance of the number of individuals migrating to this area during a fixed five-week period. *5*

2. (a) Three players A, B, C in a circle, throw a ball to the left player with probability 0·4 and to the right player with probability 0·6.

(i) Write the transition probability matrix.

(ii) Find $P^{(2)}$.

(iii) The probability that the ball is with any one player is the same, i.e. 1/3 at the start. What will be the probability that after two throws the ball will be with A ? *6*

(b) Consider a three-state Markov chain having the following transition probability matrix :

$$P = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}4 & 0{\cdot}1 \\ 0{\cdot}3 & 0{\cdot}4 & 0{\cdot}3 \\ 0{\cdot}2 & 0{\cdot}3 & 0{\cdot}5 \end{bmatrix}$$

Find the limiting probabilities of all the three states. *4*

(c) A supermarket has two exponential checkout counters, each operating at the rate μ. Arrivals are Poisson at the rate λ. The counters operate in the following way :

- One queue feeds both counters.
- One counter is operated by a permanent checker and the other by a stock clerk, who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever service is complete, and there are fewer than two customers in the system.

(i) Let P_n be the proportion of time when there are n customers in the system. Set up equations for P_n and solve them.

(ii) At what rate does the number in the system go from 0 to 1 and from 1 to 2 ?

(iii) What proportion of time is the stock clerk checking ? 5

3. (a) Let $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ and

$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}$. Obtain the distribution of

$\mathbf{Z} = \mathbf{C}\mathbf{y}$, where $\mathbf{C} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 8 & 1 \end{bmatrix}$. *5*

(b) Suppose in a city 20% individuals are smokers. It is also observed that 70% of smokers and 30% of non-smokers are males. An individual is chosen randomly from the city. What is the probability that the person is a male ? If it is observed that the person is a male, then what is the probability that he is a smoker ? *4*

(c) Customers arrive at a counter with Poisson rate 8 per hour. Service time follows exponential distribution with mean 5 minutes. Find

(i) the probability that a customer will have to wait before service;

(ii) the proportion of time the counter is idle;

(iii) the probability that the total time spent at the counter is more than 10 minutes;

(iv) the average waiting time at the counter. *6*

4. (a) Let $\mathbf{X} = X_1, X_2, X_3$ be a random vector and let the data matrix for X be

$$\begin{bmatrix} 5 & 2 & 5 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Obtain

(i) the variance-covariance matrix,

(ii) the correlation matrix. *9*

(b) The distribution of a geometric random variable describing the offspring in a branching process is given by

$p_k = pq^k,\ q = 1 - p,\ 0 < p < 1,\ k = 0, 1, 2, \ldots$.

Find the probability of extinction of the process when (i) $p = 0{\cdot}2$, (ii) $p = 0{\cdot}6$. *6*

5. (a) The variables X_1, X_2 and X_3 have the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}63 & 0{\cdot}45 \\ 0{\cdot}63 & 1 & 0{\cdot}35 \\ 0{\cdot}45 & 0{\cdot}35 & 1 \end{bmatrix}.$$

Write its factor model. *6*

(b) A Markov chain has the following transition matrix :

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}75 & 0 & 0{\cdot}25 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Determine the probabilities of ultimate return to the states and mean recurrence times of the states. Is the chain irreducible ? Give reasons for your answer. *6*

(c) Three white and three black balls are distributed in two urns in such a way that each contains three balls. The system is said to be in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, one ball is drawn from each urn and the ball drawn from the first urn is placed into the second, and conversely the ball drawn from the second urn is placed in the first urn. Let X_n denote the state of the system after the n^{th} step. Check whether $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain or not. If yes, write its transition probability matrix. If it is not a Markov chain, give the constraints so that it becomes Markovian. *3*

6. (a) Sales X_1 and profits X_2 of an industry have the following population mean and var-cov matrices

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}.$$

A sample of 10 industries gave sample mean $\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix}$. Test, at 5% level of significance, for the truthfulness of the population mean.

[You may like to use the following values :

$\chi^2_{2,0\cdot05} = 5\cdot99$, $\chi^2_{3,0\cdot05} = 7\cdot81$,

$\chi^2_{4,0\cdot05} = 9\cdot48$]. 8

(b) $\{X(t),\ t > 0\}$ is a Poisson process with parameter λ and δ_m denotes the duration from the beginning to the occurrence of the m^{th} event. Obtain the distribution of δ_m. Also, find cdf of δ_m. If $\lambda = 1$ per hour, then find the probability that duration from the start to the occurrence of the third event will be less than 2 hours. 7

7. (a) Consider the mean vectors $\mu_X = [1, 2]'$ and $\mu_Y = 3$. The var-cov matrix of $[X_1, X_2]'$ is $\Sigma_{XX} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and, $\sigma_{YY} = 14$, $\sigma_{XY} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) Fit the equation $Y = b_0 + b_1X_1 + b_2X_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean square error. 8

(b) Let 10 and 15 observations be taken for the random variables X_1 and X_2 from the populations π_1 and π_2 respectively. Let $\mu^{(1)} = [3, 1]'$, $\mu^{(2)} = [2, 1]'$ and $\Sigma^{-1} = \begin{bmatrix} 0{\cdot}15 & 0{\cdot}05 \\ 0{\cdot}05 & 0{\cdot}15 \end{bmatrix}$. Assuming equal cost and equal prior probabilities, check whether the observation $[3, 1]'$ belongs to the population π_1 or π_2. 7

8. State whether the following statements are *True* or *False*. Justify your answers with a short proof or a counter example. *5×2=10*

(a) If the transition matrix of a stochastic process is $P = \begin{bmatrix} 0\cdot3 & 0\cdot7 \\ 0\cdot7 & 0\cdot3 \end{bmatrix}$, then

$$\lim_{n \to \infty} P^n = \begin{bmatrix} 0\cdot5 & 0\cdot5 \\ 0\cdot5 & 0\cdot5 \end{bmatrix}.$$

(b) In a three-state Markov chain, all the states can be transient.

(c) If in a queuing system M/M/1, $L_s = 10$ and $W_s = 5$ minutes, then the arrival rate will be 3 per minute.

(d) The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ is a variance-covariance matrix of two random variables.

(e) In the individual replacement policy, the components are replaced at fixed time periods T, 2T, 3T, ...