No. of Printed Pages : 4

**MMT-006**

# M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

# M.Sc. (MACS)

## Term-End Examination

00722

## December, 2018

## MMT-006 : FUNCTIONAL ANALYSIS

*Time : 2 hours* *Maximum Marks : 50*

*(Weightage : 70%)*

***Note :*** *Question no.* ***6*** *is* ***compulsory****. Attempt any* ***four*** *of the remaining questions. Use of calculators is not allowed. Notations as in the study material.*

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**1.** (a) In a normed linear space X, prove that the closed ball, $\bar{u}(x, r)$ is a convex set for any $x \in X$, and $r > 0$. 3

(b) State and prove the uniform boundedness principle. 5

(c) Let $A \in BL(\mathbf{R}^3)$ be represented by the matrix $\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find $\sigma(A)$. 2

2. (a) Prove that $(C_{00}, ||\cdot||_p)$ is not a Banach Space for any $1 \leq p \leq \infty$. *3*

(b) State and prove Bessel's inequality for a finite orthonormal set in an inner product space. *3*

(c) Let $X = \mathbf{C}^2$ and $\alpha_1, \alpha_2, \in \mathbf{C}$. Let $A : X \to X$ be given by

$$A(x_1, x_2) = (\alpha_1 x_1, \ \alpha_2 x_2), \ x_1, x_2 \in \mathbf{C}.$$

Prove that

(i) A is normal

(ii) A is self-adjoint $\Leftrightarrow \alpha_1, \alpha_2 \in \mathbf{R}$

(iii) A is unitary $\Leftrightarrow |\alpha_1| = 1 = |\alpha_2|$ *4*

3. (a) Let Y be a closed subspace of a normed linear space X. Show that X is complete $\Leftrightarrow$ Y and $\frac{X}{Y}$ are complete. *5*

(b) Define a reflexive normed linear space. Prove that if a normed space X is reflexive then its dual space is also reflexive. What about the converse ? Justify your answer. *5*

4. (a) Prove that the dual space of $l^\infty$ contains a proper subspace which is linearly isometric to $l^1$. *4*

(b) Define $T : (C([0, 1]), \|\cdot\|_{\infty}) \to \mathbf{C}$ by

$$T_f = \int_0^1 f(t)\, dt .$$

Show that T is bounded and find $\| T \|$. *3*

(c) Let $X = L^2[0, 1]$ with $\|\cdot\|_2$ given by

$$\| f \|_2 = \left( \int_0^1 | f(t) |^p \, dt \right)^{1/p}, f \in X.$$

Let f and g be functions given by

$f(t) = t$, $g(t) = 1 - t$, $\forall\, t \in [0, 1]$. Show that

$\| f + g \|_2^2 + \| f - g \|_2^2 = 2(\| f \|_2^2 + \| g \|_2^2 )$. *3*

**5.** (a) Let Y be a subspace of a normed linear space X and let $a \in X$ such that $a \notin \overline{Y}$. Show that $\exists\, f \in X'$ such that $f(y) = 0\ \forall\, y \in Y$, $f(a) = \text{dist}(a, \overline{Y})$ and $\| f \| = 1$. *3*

(b) Let $x_n(t) = t^n$ for $n = 0, 1, 2, \ldots, -1 \le t \le 1$. Prove

(i) $\{x_0, x_1, x_2, \ldots\}$ is linearly independent in $L_2([-1, 1])$.

(ii) Apply the Gram-Schmidt process on $\{x_0, x_1, x_2, \ldots\}$ to determine the first three vectors in the orthonormal set. *4*

(c) Let $X$ be a normed space, $z \in X$ and $f \in X'$. Show that $T : X \to X$ defined by $T(x) = f(x)z$, $x \in X$ is linear and compact. *3*

**6.** Are the following statements *True* or *False* ? Justify your answers. *5×2=10*

(a) If $X$ is a normed linear space over $\mathbf{R}$ and suppose $f(x) = 0 \; \forall \; f \in X'$, then $x = 0$.

(b) On an infinite dimensional normed linear space every linear function is continuous.

(c) If $T : X \to Y$ is linear, continuous and open, then $T$ is subjective.

(d) In an inner product space if $x \neq 0$, then $\langle x, y \rangle \neq 0$ for some $y$.

(e) Any two separable Hilbert spaces need not be isometric.