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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00422

Term-End Examination

December, 2018

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. **1** is **compulsory**. Attempt any **four** questions from questions no. **2** to **6**. Calculators are **not** allowed.*

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1. State whether the following statements are *True* or *False*. Give reasons for your answers. *5×2=10*

(a) The set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is neither dense nor nowhere dense in $(\mathbf{R}, d)$.

(b) $(1, 0, -1)$ is a critical point of the function $f(x, y, z) = 1 + |x| + |y| + |z|$.

(c) Every closed and bounded subject of a metric space is compact.

(d) If a set E has finite measure, then $L^1(E) \subset L^2(E)$.

(e) Let $E_n = \left[0, \frac{1}{n}\right]$, then $m^*(\cap E_n) = 0$.

2. (a) Show that there is a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(x, y) = 1 \text{ if } (x+1)^2 + (y-1)^2 \le \frac{1}{2} \text{ and}$$

$$f(x, y) = 2 \text{ if } (x+1)^2 + (y+1)^2 \le \frac{1}{2}.$$ 2

(b) Find the directional derivative of the function

$$f(x, y, z, w) = (x^2 - y^2, 2xy, zx, z^2w^2x^2)$$

at the point (1, 2, – 1, – 2) in the direction of (2, 1, – 2, – 1). 3

(c) Let $|f_n|$ be a monotonically increasing sequence of non-negative measurable functions converging to a function f. Is f measurable ? Justify your answer. Prove also that

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$ 5

3. (a) Define component of a metric space. Show that a metric space can be written as a disjoint union of its components and that each connected subset of the space intersects only one of them. 5

(b) Suppose f is a non-negative measurable function. Prove that

$$\int_{\mathbf{R}} f \, dm = 0 \text{ if and only if } f = 0 \text{ a.e.}$$

Is this result true if the non-negative condition is dropped ? Justify your answer. 5

4. (a) Let X, Y be metric spaces and $f : X \to Y$ a map. Prove that f is continuous if and only if for every open set $V \subset Y$, the set $f^{-1}(V)$ is open in X. 4

(b) For the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$f(x, y, z) = (x^2y, y^2z + z^2, -2x),$$

find $f'\left(1, 1, -\frac{1}{2}\right)$. 3

(c) Define the Fourier transform of the function $f \in L^1(\mathbf{R})$. Find the Fourier transform of the function $f = \chi_E$ where $E \left[0, \frac{1}{2}\right]$. 3

5. (a) Let (X, d) be a metric space with $X \neq \{0\}$ and $x \in X$ and $0 < r < s$. Show that $B[x, r] \subseteq B[x, s]$. Give an example to show that it is possible to have $B[x, r] = B[x, s]$. 3

(b) Obtain the 2^{nd} order Taylor's series expansion of the function

$$f(x_1, x_2) = x_1^2 x_2^4 + x_1^3 e^{x_2} \text{ at } (1, 1).$$ 4

(c) When is a function said to satisfy the Lipschitz condition ? If f is an integrable function on $[-\pi, \pi]$, which satisfies the Lipschitz condition on $[-\pi, \pi]$, prove that

$$\lim_{n \to \infty} S_n(f; \theta) = f(\theta).$$ 3

6. (a) Show that every convergent sequence in a metric space (X, d) is a Cauchy sequence. Is the converse true ? Justify your answer. 4

(b) Let $f = (f_1, f_2)$ be a vector valued function from $\mathbf{R}^5$ to $\mathbf{R}^2$ where f_1, f_2 are defined by

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.$$

Prove that f defines a unique function $g : \mathbf{R}^3 \to \mathbf{R}^2$ in a neighbourhood of (3, 2, 7) such that $g(3, 2, 7) = (0, 1)$. 4

(c) Prove that the system

$$g(t) = \Re(f(t)) = \int_{-\infty}^{2t} f(\tau)\, d\tau$$

is a causal system. 2