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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

00551

Term-End Examination

December, 2017

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is* **compulsory**. *Answer any* **six** *questions from questions no. 1 to 7. Use of scientific, non-programmable calculator is allowed. All the symbols used have their usual meaning.*

1. (a) Let X follows $N_3(\mu, \Sigma)$ with

$$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Examine the independence of the following : *8*

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $\dfrac{X_1 + X_2}{2}$ and X_3

(b) On the basis of sales (X_1) and profits (X_2) of 10 industries, the following sample mean was obtained :

$$\overline{X} = \begin{pmatrix} \overline{X}_1 \\ \overline{X}_2 \end{pmatrix} = \begin{pmatrix} 33 \\ 7 \end{pmatrix}$$

Expected mean vector and variance-covariance matrix is

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}, \text{ respectively.}$$

Test whether the sample confirms the truthness of mean vector at 5% level of significance. [You may use the values : $\chi^2_{2,\,0\cdot05} = 5\cdot99$, $\chi^2_{9,\,0\cdot05} = 16\cdot92$] 7

2. (a) Let a random variable X have the following probability distribution :

X	– 2	– 1	0	1	2
f(x)	0·15	0·2	0·3	0·2	0·15

Assume $Y = X^2$.

(i) Obtain joint probability distribution of X and Y.

(ii) Obtain covariance of X and Y.

(iii) Examine the independence of X and Y. 7

(b) A post-office has two counters. The first counter handles speed post and the second counter handles all the other tasks. Service times of both the counters follow exponential law with mean 4 minutes. Customers arrive at the first counter at the rate of 12 per hour and at the second counter at the rate of 13 per hour in a Poisson way.

(i) Obtain the average waiting time in queue at both the counters.

(ii) If both the counters are allowed to handle all the tasks, then obtain the average waiting time in queue by the customers. *8*

3. (a) Obtain a lower triangular square root of the matrix $\begin{bmatrix} 9 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 6 \end{bmatrix}$. *6*

(b) Weather data on four variables, x_1 = average minimum temperature, x_2 = average relative humidity in a span of 8 hours, x_3 = average relative humidity in a span of 14 hours and x_4 = total rainfall in 17 years from 1970 to 1986 are provided in the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 17{\cdot}02 & -4{\cdot}12 & 1{\cdot}54 & 5{\cdot}14 \\ & 7{\cdot}56 & 8{\cdot}50 & 54{\cdot}82 \\ & & 15{\cdot}75 & 92{\cdot}95 \\ & & & 903{\cdot}87 \end{bmatrix}$$

The eigenvalue-eigenvector pairs of Σ are

$\lambda_1 = 916{\cdot}9$ $\quad a_1 = (0{\cdot}006, 0{\cdot}061, 0{\cdot}103, 0{\cdot}993)$

$\lambda_2 = 18{\cdot}4$ $\quad a_2 = (0{\cdot}955, -0{\cdot}296, 0{\cdot}011, 0{\cdot}012)$

$\lambda_3 = 7{\cdot}9$ $\quad a_3 = (0{\cdot}141, 0{\cdot}485, 0{\cdot}855, -0{\cdot}119)$

$\lambda_4 = 1{\cdot}0$ $\quad a_4 = (0{\cdot}260, 0{\cdot}820, -0{\cdot}509, 0{\cdot}001)$

(i) Obtain principal components.

(ii) Verify that the total variance of principal components is the same as total variance of original variables.

(iii) Obtain proportion of total variation explained by the first component and the first two components.

(iv) Obtain the values of the first two components if the actual data of the first year be
$x_1 = 25, x_2 = 86, x_3 = 66, x_4 = 186{\cdot}49.$ *9*

4. (a) The P-matrix of a Markov chain is given below. Draw its transition diagram and obtain P^n and its limiting value for large n. *10*

$$P = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}25 & 0{\cdot}5 & 0{\cdot}25 \\ 0 & 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$$

(b) The offspring distribution in a branching process is

$$p_0 = q, p_1 = 0, p_2 = p, p_r = 0, r \geq 3.$$

Discuss the probability of extinction for this process. 5

5. (a) A plant producing memory chips has 3 assembly lines. Line 1 produces 30% of the chips with a defective rate 3%, line 2 produces 50% of the chips with a defective rate 4% and rest of the chips produced by line 3 have a defective rate 2%. A chip is selected from the plant.

(i) Find the probability that the selected chip is defective.

(ii) Given that the chip is defective, find the probability that the chip was produced by line 2. 5

(b) Let Σ be the variance-covariance matrix and **R** be the correlation matrix of a random vector **X**. Let T be a diagonal matrix with the elements being standard deviation of the respective variables. Show that

$$\mathbf{R} = \mathbf{T}^{-1} \Sigma \mathbf{T}.$$ 4

(c) The owner of a chain of five stores wishes to forecast net profit with the help of next year's projected sales of food and non-food items. The data about current year's sales of food items, sales of non-food items as also net profit for all the five stores are available as follows :

Store No.	Net Profit y (₹ in lakhs)	Sales of food items x_1 (₹ in lakhs)	Sales of non-food items x_2 (₹ in lakhs)
1	20	9	6
2	14	7	7
3	7	11	7
4	12	6	10
5	8	5	11

Assuming a linear regression model

$$y_i = b_0 + b_1 x_{1i} + b_2 x_{2i} + e_i,$$

where i = 1, 2, 3, 4, 5.

Find

(i) the least square estimates $\hat{b}$,

(ii) the residuals $\hat{e}$,

(iii) the residual sum of squares for the model. *6*

6. (a) Let $X \sim N_3(\mu, \Sigma)$ where $\mu = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and

$$\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Obtain the following : *8*

(i) Marginal distribution of $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$

(ii) Distribution of $Z = X_1 - 2X_2 + X_3$

(iii) Conditional density of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, given X_3

(iv) $r_{12 \cdot 3}$

(b) A machine shop needs a certain kind of machine regularly. Whenever a machine fails it is replaced immediately. Assume life time of machines follows uniform distribution in the interval [5, 10] years. Find the rate of replacement in a long time. *4*

(c) If $\{X(t) : t > 0\}$ is a Poisson process with rate λ and S_m denotes the duration from start to the occurrence of m^{th} event, obtain the distribution of S_m. If $\lambda = 1$ per hour, then find the probability that the duration from start to the occurrence of third event will be less than 2 hours. *3*

7. (a) On the basis of 50 observations on 4 variables, the factor loadings of the first two factors obtained through factor analysis are :

Variables	Factors I	Factors II	Communality
X_1	0·697	0·476	0·712
X_2	0·748	0·445	0·758
X_3	0·831	0·350	0·813
X_4	0·596	0·648	0·775
Sum of squares	2·091	0·967	
Variance summarized	0·523	0·242	0·765

(i) Write linear equations for all the factors.

(ii) Interpret the loading coefficients, variance summarized and communality values. *6*

(b) If interoccurrence time in a renewal process follows geometric distribution with parameter p, show that number of occurrences N_n in n time follows binomial distribution. *3*

(c) Let the life times X_1, X_2, ... be i.i.d. exponential random variables with parameter $\lambda > 0$. Let $T > 0$ and age replacement policy is to be employed.

(i) Find mean.

(ii) If each replacement cost $C_1 = 3$ and extra cost $C_2 = 4$, then find the long run average cost per unit time. *6*

8. State whether the following statements are *True* or *False*. Justify your answers. *10*

(a) The probability density function of a random variable lies between 0 and 1.

(b) A state in the Markov chain is transient, if the probability of ultimate return to the state is less than 1.

(c) If $p_0 = 1$ in a branching process, then the probability of ultimate extinction of the process will be smaller than 1.

(d) Posterior probabilities obtained from Bayes' theorem are larger than respective prior probabilities.

(e) T^2 statistics is invariant to the change of origin but not invariant to the change of scale.