M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00774

Term-End Examination

December, 2016

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is **not** allowed. All the symbols used have their usual meaning.*

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1. (a) Arrivals at a counter in a bank occur in accordance with Poisson law having average rate of 10 per hour. The service time of a customer follows exponential law with mean time 5 minutes. Find the following : *5*

   (i) Average number of customers at the counter including the one that is being served.

   (ii) Average time required in getting service to a new arrival.

(iii) Probability that a new arrival finds the counter empty when he/she arrives.

(iv) Probability that a new arrival finds at least five customers at the counter.

(b) A bag contains 4 red and 6 black balls. Two red balls and one black ball were marked for superior quality. One ball was chosen from the bag and the ball was marked. What is the probability that the ball was red ? *4*

(c) Consider the random vector $X' = (X_1, X_2, X_3)$ having the following covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 1 & 0{\cdot}25 \\ 0{\cdot}5 & 0{\cdot}25 & 1 \end{bmatrix}$$

Write its factor model with one underlying factor. *6*

2. (a) The joint probability density function of continuous random variables X and Y is given below :

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the marginal probability density functions of X and Y.

(ii) Test independence of X and Y.

(iii) Compute $P[X > 0{\cdot}1]$.

(iv) Find the conditional probability density function of X, given $Y = 0{\cdot}3$. 7

(b) A sample of 10 industrial corporations observed for their sales (X_1) and profits (X_2) has provided the following information :

$$\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 0{\cdot}01 & -0{\cdot}03 \\ -0{\cdot}03 & 0{\cdot}25 \end{bmatrix}$$

Test whether the average industrial sales and profits may be accepted at 5% level of significance.

Given $\mu = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$. [You may like to use the values $F_{2,\,8,\,0{\cdot}05} = 4{\cdot}46$, $F_{2,\,10,\,0{\cdot}05} = 19{\cdot}40$] 6

(c) Consider a renewal process whose mean-value function is given by $m(t) = 2t$, $t \geq 0$. What is the distribution of the number of renewals occurring by time 10 ? 2

3. (a) Let X be $N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Check the independence of the following random variables and justify your answer : 8

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $(X_1 + X_2)$ and X_3

(b) Suppose 10 and 15 observations are made on random variables X_1 and X_2, respectively, from two populations π_1 and π_2. Let $\mu^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mu^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\Sigma^{-1} = \begin{bmatrix} 0{\cdot}15 & 0{\cdot}05 \\ 0{\cdot}05 & 0{\cdot}15 \end{bmatrix}$.

Considering equal costs and equal prior probabilities, check whether the observation [3, 1] belongs to population π_1 or π_2. 7

4. (a) In a two-state simple weather model, the probability of a dry day following a rainy day is 0·2 and the probability of a rainy day following a dry day is 0·1.

(i) Write the transition probability matrix P for this Markov chain.

(ii) Find $P^{(3)}$.

(iii) If the probability of a dry day on 15^{th} July, is 0·3, then find the probability that 18^{th} July will be a dry day. *8*

(b) Define a renewal process. If the renewal process $\{N_t, t = 0, 1, 2, ...\}$ is a negative binomial process, then obtain its renewal function. *4*

(c) Suppose that p(x, y), the joint probability mass function of X and Y, is given by $p(1, 1) = 0{\cdot}5$, $p(1, 2) = 0{\cdot}1$, $p(2, 1) = 0{\cdot}1$, $p(2, 2) = 0{\cdot}3$. Calculate the probability mass function of X, given that Y = 1. *3*

5. (a) For a Poisson process $\{X(t) : t \geq 0\}$, find the probability that there are k events in time t, given that there are k + n events in time t + s. *4*

(b) Let $X = (X_1, X_2, X_3)'$ follow $N_3(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

(i) Write down the marginal distributions of X_2 and X_3.

(ii) Write down the marginal distribution of $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$.

(iii) Obtain the conditional distribution of $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$, given $X_1 = 3$. *8*

(c) Write any three applications of Conjoint Analysis. *3*

6. (a) Let $X' = (X_1, X_2, X_3)$ be a random vector and the following be the data matrix of X :

$$X = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Find

(i) variance-covariance matrix,

(ii) correlation matrix. *9*

(b) In a branching process, the offspring distribution is given as

$$p_k = \binom{n}{k} p^k q^{n-k} \,;\, k = 0, 1, 2, \ldots, n,$$

$$q = 1 - p,\ 0 < p < 1.$$

Find the probability of ultimate extinction of the process when

(i) $n = 2,\ p = 0{\cdot}3$

(ii) $n = 2,\ p = 0{\cdot}8$ *6*

7. (a) A Markov chain has the following transition matrix :

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Find the stationary distributions of the chain.

(ii) How many distributions are possible ?

(iii) How many closed sets are there in the chain ? *6*

(b) Consider a system of two servers where customers from outside the system arrive at Server 1 at a Poisson rate 4 and at Server 2 at a Poisson rate 5. The service rates of Servers 1 and 2 are 8 and 10, respectively. A customer upon completion of service at Server 1 is equally likely to go to Server 2 or to leave the system, whereas, a departure from Server 2 will go 25 percent of the time to Server 1 and will depart the system otherwise. Determine the limiting probabilities. *9*

8. State whether the following statements are *True* or *False*. Justify your answers. *2×5=10*

(a) If P is a 4×4 transition probability matrix, then the sum of all elements of P is 4.

(b) If at a doctor's chamber the arrival rate is larger than departure rate, then the number of patients in the chamber will not change in the visiting hour.

(c) Bayes' theorem improves the probability of an event using information on the happening of other event.

(d) Principal components of a set of variables do not depend upon the scales used to measure the variables.

(e) In age replacement policy, the component is replaced at fixed time points T, 2T, 3T, ...