No. of Printed Pages : 8 **MMT-008**

# M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

01254

## M.Sc. (MACS)

## Term-End Examination

## December, 2015

## MMT-008 : PROBABILITY AND STATISTICS

*Time : 3 hours* *Maximum Marks : 100*

*(Weightage : 50%)*

---

***Note :*** *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is **not** allowed.*

---

1. (a) Consider a closed queuing network with three queues with exponential services. Service rates for these queues are $\mu_1$, $\mu_2$ and $\mu_3$. The total number of clients in the queue is N = 3. Routing probabilities between the queues are defined by the following matrix :

$$[p_{ij}] = \begin{bmatrix} 0 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \end{bmatrix}$$

(i) Draw the queuing system.

(ii) Find the steady state probabilities.

(iii) If $\mu_1 = \mu_2 = \mu_3 = \mu$, then find the average number of customers in steady state and also find the average time spent by the customers in each queue. 7

(b) A fair die is tossed and its outcome is denoted by X, i.e.

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

After that, X independent fair coins are tossed and the number of heads obtained is denoted by Y.

Find : 8

(i) P[Y = 4]

(ii) P[X = 5 | Y = 4]

(iii) E(Y)

(iv) E(XY)

2. (a) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is given by

$$\sum = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$ 6

(b) Let $X \sim N_4(\mu, \Sigma)$ with

$$\mu = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 3 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 4 & 2 & -3 & 4 \\ 2 & 4 & 0 & 1 \\ -3 & 0 & 4 & -2 \\ 4 & 1 & -2 & 8 \end{bmatrix}.$$

Let Y and Z be two partitioned subvectors of X such that $Y' = [X_1, X_2]$ and $Z' = [X_3, X_4]$.

Find : 9

(i) $E(Y/Z)$

(ii) $Cov(Y, Z)$

(iii) $r_{12.34}$

3. (a) The transition probability matrix P of a Markov chain with states Sunny (S), Cloudy (C) and Rainy (R) in a simple weather model is given below :

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0{\cdot}7 & 0{\cdot}2 & 0{\cdot}1 \\ 0{\cdot}2 & 0{\cdot}5 & 0{\cdot}3 \\ 0{\cdot}1 & 0{\cdot}5 & 0{\cdot}4 \end{bmatrix} \end{array}$$

Also the initial probability distribution of the states is $(0{\cdot}5 \quad 0{\cdot}3 \quad 0{\cdot}2)$.

(i) What is the probability that starting from initial day, all the three successive days will be cloudy ?

(ii) Obtain the probability distribution of the weather on the second day. 5

(b) A radioactive source emits particles at a rate of 5 per minute according to Poisson law. Each particle emitted has probability 0·6 of being recorded. What is the probability that in 4 minutes 10 particles will be recorded ? What is the mean and variance of the number of particles recorded ? 5

(c) Let the lifetimes $X_1$, $X_2$, ... be i.i.d. exponential random variables with parameters $\lambda > 0$ and $T > 0$. Age replacement policy is to be employed.

(i) Find $\mu^T$.

(ii) If the replacement cost $C_1 = 3$ and extra cost $C_2 = 4$, then find the long run average cost per unit time. 5

4. (a) Let $\{X_n, n = 1, 2, ...\}$ be i.i.d. geometric random variables with the probability mass function

$$P(X_n = i) = (1 - p)p^{i-1}, i = 1, 2, 3, ...$$

Find the renewal function of the corresponding renewal process. 7

(b) Let $\bar{X} = \{X_1, X_2, X_3\}$ be a random vector and X be the data matrix given below :

$$X' = \begin{bmatrix} 5 & 2 & 5 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Find : 8

(i) Variance – Covariance matrix $\Sigma$

(ii) Correlation matrix R

5. (a) On the basis of past experience about the sales ($X_1$) and profits ($X_2$) the population mean vector and variance – covariance matrix for the industry was as given below :

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}$$

From a sample of 10 industries the sample mean vector was found as $\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix}$. Test at 5% level of significance whether the sample confirms the truthfulness of the industry claim of population mean vector. [You may like to use the following values : $x^2_{2, 0 \cdot 05} = 5 \cdot 99$, $x^2_{3, 0 \cdot 05} = 7 \cdot 81$] 7

(b) Suppose $X = (X_1, X_2, X_3)'$ be distributed as a trivariate normal distribution, $N_3(\mu, \Sigma)$, where

$$\mu = (2, 1, 3)', \quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Find the distribution of

$$u = \begin{bmatrix} X_1 - X_3 \\ X_1 + X_3 - 2X_2 \end{bmatrix}.$$ 8

**6.** (a) In a Branching process, the offspring distribution is given as

$$P_k = {}^nC_k \, p^k \, q^{n-k}, \; k = 0, 1, 2, \ldots, n$$
$$q = 1 - p,$$
$$0 < p < 1.$$

Find the probability of ultimate extinction of the process given that

(i) $n = 2, p = 0{\cdot}2$,

(ii) $n = 2, p = 0{\cdot}8$. 7

(b) If $\{ X(t) \; ; \; t > 0 \}$ is a Poisson process with rate $\lambda$ and $S_m$ denotes the duration from start to the occurrence of $m^{th}$ event, obtain the distribution of $S_m$. If $\lambda = 1$ per hour, then find the probability that the duration from start to the occurrence of third event will be less than 2 hours. 8

7. (a) Suppose that the random variables X and Y have the following joint p.d.f. :

$$f(x, y) = \begin{cases} x + y \,; & 0 < x < 1,\ 0 < y < 1 \\ 0 \,; & \text{otherwise} \end{cases}$$

(i) Find the conditional p.d.f. of X given $Y = y$.

(ii) Check independence of X and Y.

(iii) Compute $P\left[0 < X < \frac{1}{3} \,\middle|\, Y = \frac{1}{2}\right]$. *7*

(b) Let X denote the data matrix for a random sample of size 3 from a bivariate normal population, where

$$X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}.$$

Test the hypothesis $H_0 : \mu = (9, 5)'$ at 5% level of significance. [You may like to use the following values : $F_{2,\,1,\,0\cdot05} = 18\cdot51$, $F_{3,\,2,\,0\cdot05} = 9\cdot55$] *8*

8. State whether the following statements are *true* or *false*. Give a short proof or counter-example in support of your answer. *10*

(a) If $f_{ij} < 1$ and $f_{ji} < 1$, then i and j do not intercommunicate.

(b) If P(s) is the generating function of the random variable X, then the generating function of 2X + 1 is 2P(s) + 1.

(c) If $\{n(t), t > 0\}$ is a Poisson process with rate $\lambda$, then $E[n(t+s) - n(s)] = \lambda t$.

(d) The quadratic form $Q = 2x_1^2 - 3x_2^2 - 6x_1x_2$ is negative definite.

(e) If $X \sim N_p(\mu, \Sigma)$, then the linear combinations of the components of X are normally distributed.

---