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MMT-006

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

00483

December, 2015

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*
(Weightage : 70%)

Note : *Question no. 1 is **compulsory**. Attempt any **four** of the remaining questions.*

1. State whether the following statements are *true* or *false*. Give a brief justification, with a short proof or a counter-example : *5×2=10*

   (a) If X is a normed linear space, $x, y \in X$ and $\| x \| = 1 = \| y \|$, then $\| x + y \| < 2$.

   (b) The map $l' \to l'$, $(x_n) \mapsto ((-1)^n x_n)$ is open.

   (c) The space $(\mathbf{R}^2, \|\cdot\|_\infty)$ is not reflexive.

   (d) The linear operator $A : \mathbf{C}^2 \to \mathbf{C}^2$, given by $A(z_1, z_2) = (z_1, -z_2)$ is a normal operator.

   (e) $C'[0, 1]$ is not complete under sup norm.

2. (a) On $\mathbf{R}^3$, prove that $\| x \| = |x_1| + \sqrt{x_2^2 + x_3^2}$ defines a norm. Find the constants $\alpha, \beta > 0$ such that $\alpha \| x \|_2 \leq \| x \| \leq \beta \| x \|_2$, $x \in \mathbf{R}^3$. 5

(b) (i) Define the spectrum and eigen spectrum for a bounded linear operator defined over a Banach space.

(ii) Let $X = C_0$ and $A : X \to X$ be the diagonal operator defined as follows :

If $x = \{x_j\}$, then

$$(Ax) = \left\{\frac{x_j}{j}, j \in \mathbf{N}, x \in X\right\}.$$

Show that

$$\sigma_e(A) = \left\{\frac{1}{j} \,\middle|\, j \in \mathbf{N}\right\}.$$ *5*

3. (a) Let X be a real normed linear space and let Y be a proper, closed, linear subspace of X. Show that, for any $x_0 \in X \setminus Y$, there exists a bounded linear functional $f : X \to \mathbf{R}$ such that f is 0 on Y and $f(x_0) = 1$. *3*

(b) Define the transpose A' of a bounded linear operator A and prove that $\| A' \| = \| A \|$. *3*

(c) If A is a positive operator on a Hilbert space, show that I + 2A is invertible. *4*

4. (a) Let A be an operator on $l^2(\mathbf{C})$ defined as follows :

If $(x_n) \in l^2(\mathbf{C})$, $Ax = (e^{in} x_n)$.

Show that A is a unitary operator. *3*

(b) If M and N are closed linear subspaces of a Hilbert space, show that

$$M \cap N = (M^{\perp} + N^{\perp})^{\perp}.$$ 3

(c) State closed graph theorem. Show that the theorem may not hold, if the normed linear spaces involved are not Banach spaces. 4

5. (a) Let X be a Banach space and Y be a normed linear space. If A_n, $A \in BL(X,Y)$ and $A_n x \to Ax$ for all x, then show that the convergence is uniform on any totally bounded set. 4

(b) (i) Prove that

$$M = \{ f \in L^2 [0, 1] : \int_0^1 f(t)\, dt = 0 \}$$

is a closed linear subspace of $L^2 [0, 1]$.

(ii) Show that $P(f) = f - \int_0^1 f(t)\, dt$ is a projection of $L^2 [0, 1]$ on M. 6

6. (a) Let T be a linear operator on a normed linear space X. Show that, if T is continuous at 0, it is continuous on X. Is T uniformly continuous on X ? Justify your answer. *3*

(b) Find the norm of the operator A defined on l' as follows : *2*

$$\text{If } x = x_n \in l', \quad Ax = \frac{x_n}{2n}.$$

(c) Give an example, with justification, of an orthonormal sequence on l^2. Further, show that if $\{ l_n \}$ is an orthonormal sequence on a Hilbert space and α_n are scalars such that $\Sigma \, |\alpha_n|^2 < \infty$, $\Sigma \, \alpha_n e_n$ is convergent. *5*

7. (a) Prove that a normed linear space is separable, if its dual is separable. *4*

(b) Let A be a bounded linear operator on a Hilbert space. Show that if R(A) is closed, then so is $R(A^*)$. *4*

(c) Give an example of a Banach space that is not a Hilbert space. *2*