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MMT-004

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

December, 2015

MMT-004 : REAL ANALYSIS

Time : 2 hours *Maximum Marks : 50*

(Weightage : 70%)

Note : *Question no. 1 is* ***compulsory****. Attempt any* ***four*** *questions from questions no. 2 to 7. Calculators are* ***not*** *allowed.*

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1. State whether the following statements are *True* or *False*. Give reasons for your answer. *5×2=10*

(a) Continuous image of a Cauchy sequence is a Cauchy sequence.

(b) Every connected metric space is path connected.

(c) The function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$f(x, y) = (x^2 - y^2, 2xy)$ is invertible.

(d) The projection map defined on $(\mathbf{R}^2, d)$, where d is the usual metric, is continuous.

(e) Every Lebesgue integrable function is Riemann integrable.

2. (a) Let $X = C[0, 1]$. For $f, g \in C[0, 1]$ define

$$d : X \times X \to \mathbf{R} \text{ by}$$

$$d(f, g) = \int_0^1 | f(t) - g(t) | \, dt,$$ where the integral is the Riemann integral. Show that d is a metric on X. Find d(f, g), where $f(x) = 2x$ and $g(x) = x^2$, $x \in [0, 1]$. *5*

(b) Show that every compact set in a metric space is closed and bounded. Is the converse true ? Justify your answer. *5*

3. (a) Find the directional derivative of the function $f : \mathbf{R}^4 \to \mathbf{R}^3$ defined by $f(x, y, z, w) = (x^2y, xyz, x^2 + y^2 + zw^2)$ at $a = (1, 2, -1, -2)$ in the direction

$$V = (0, 1, 2, -2).$$ *4*

(b) Show that the outer measure of a countable set is zero. Also compute $m^* (\mathbf{R} - \mathbf{Q})$, where $\mathbf{Q}$ is the set of all rational numbers. *3*

(c) Find the Fourier series for the function

$$f(t) = \begin{cases} -6-t, & -6 \le t \le -3 \\ t, & -3 \le t \le 3 \\ 6-t, & 3 \le t \le 6. \end{cases}$$ 3

4. (a) Find the extreme values of the function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

subject to the constraint

$$4x_1 + x_2^2 + 2x_3 = 14,$$

$$x_1, x_2, x_3 \ge 0.$$ 4

(b) Let (X, d) be a metric space, and A be the subset of X. Show that

(i) $\overline{A} = \text{int}\, A \cup \text{bdy}\, A$.

(ii) A is closed, if and only if $A \ge \text{bdy}\,(A)$. 3

(c) State a sufficient condition for which the convolution integral of two Lebesgue integrable functions exists for all $x \in \mathbf{R}$. Show that for the following functions, the convolution integral does not exist : 3

$$f(t) = \begin{cases} \dfrac{1}{\sqrt{t}}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \le 0 \text{ or } t \ge 1 \end{cases}$$

$$g(t) = \begin{cases} \dfrac{1}{\sqrt{1-t}}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \le 0 \text{ or } t \ge 1 \end{cases}$$

5. (a) Let $h(t) = e^{-2t} u(t)$. Then find the system response to the input function

$$f(t) = \sum_{k=-2}^{2} \left(\frac{1}{2}\right)^k e^{i3kt}.$$ 3

(b) Let (X, d) be a metric space and $a \in X$ be a fixed point of X. Show that the function $f_a : X \to \mathbf{R}$ given by $f_a(x) = d(a, x)$ is uniformly continuous on X. 4

(c) Expand the function $f(x) = x$, if $0 \le x \le \pi$ as a Fourier sine series. 3

6. (a) State the implicit function theorem for $\mathbf{R}^3$. Let $f = \mathbf{R}^3 \to \mathbf{R}$ defined by

$$f(x, y, z) = x^2 + y^3 - xy \sin z.$$

Show that the equation $f(x, y, z) = 0$ defines a unique continuously differentiable function g in a neighbourhood of the point $(1, -1, 0)$ such that $g(1, -1) = 0$. 5

(b) Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \xrightarrow{\text{onto}} Y$ be a continuous mapping. If X is connected, then show that Y is connected. 2

(c) Define a time invariant system. Check whether the system $\mathbf{R} : f \to g$ given by

$$g(t) = (\mathbf{R}f)(t) = \int_{-\infty}^{t} f(c)\, dc, f \in L'(\mathbf{R})$$ is a time invariant system. 3

7. (a) State Baire Category theorem and use it to check whether **Q** with usual metric is complete. *3*

(b) Use the Dominated Convergence theorem to find $\lim\limits_{n \to \infty} \int_1^{\infty} f_n(x)\, dx$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$. *3*

(c) Show that in a metric space, a finite intersection of open sets is open. Give an example to show that an arbitrary intersection of open sets may not be open. *4*