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# M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

**Term-End Examination** 00780

**December, 2013**

## MMT-008 : PROBABILITY AND STATISTICS

*Time : 3 hours* *Maximum Marks : 100*

*(Weightage : 50%)*

*Note : Question number 8 is **compulsory**. Answer **any six** questions from question number **1** to **7**. Use of calculator is **not** allowed.*

**1.** (a) Due to a faulty pipe, a certain road gets waterlogged with probability 0.1 on a normal day and with probability 0.7 on a rainy day. It rains approximately half the number of days in July. Ms. Rajni generally reaches office on time 90% of the days if the road is not water logged. But in case of water logging she reaches office on time only 50% of the days. If on a particular day in July, Ms. Rajni was late to office, what is the probability that the road was waterlogged on that day ? **6**

(b) Obtain a spectral decomposition of the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ **6**

(c) Consider the mean vector $\mu_x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and 3

$\mu_y = 2$, and the covariance matrices of $x_1$, $x_2$ and $y$ are

$$\Sigma_{xx} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \; \sigma_{yy} = 9, \; \sigma_{xy} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation.

**2.** (a) Consider a Markov chain $\{X_1, X_2 \ldots.\}$ with state space $S = \{1,2,3\}$, initial distribution $\pi = (0.1 \; 0.4 \; \; 0.5)$ and transition probability matrix P given by 7

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

(i) Find the probability distribution of $X_2$.

(ii) Find prob.$(X_1 = X_2 = X_3 = X_4 = 2)$.

(iii) Find prob. $(X_3 = X_4 = 3)$.

(b) Let $X \sim N_3\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}\right)$. 8

Let $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ where $Y_1 = 2X_1 + X_3$ and $Y_2 = X_2 + 2X_3$.

(i) Find the distribution of Y.

(ii) Find the conditional distribution of $Y_1$ given $Y_2 = 5$.

3. (a) Consider a branching process with offspring distribution $\{p_j\}$ given by **7**

$$P_j = \begin{cases} \frac{1}{3} & j=0, 1, 3. \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability of extinction.

(b) Let the data matrix from a random sample of size n=3 from a bivariate normal population be **8**

$$X = \begin{bmatrix} 3 & 7 \\ 8 & 9 \\ 4 & 8 \end{bmatrix}$$

(i) Evaluate $T^2$ for testing $H_0 : \mu = [7,7]$.

(ii) Test $H_0$ against $H_1 : \mu \neq [7\ 7]$ at level of significance $\alpha = 0.10$.

(You may use the following tabulated values of F. $F_{2,1}(0.1) = 49.50$, $F_{1,2}(0.1) = 39.86$, $F_{2,3}(0.1) = 5.46$, $F_{3,2}(0.1) = 9.16$.)

4. (a) Aman goes to college by bus. He can use either bus route A or route B to do so. Buses plying on route A arrive in a Poisson fashion with mean $\lambda$ per hour ($\lambda>0$), while for route B buses, the time between two consecutive arrivals is exponential with mean $\frac{1}{2\lambda}$ hours. **6**

If buses on routes A and B run independently of each other, and if Aman boards the first bus (A or B) that arrives, what is the probability that Aman has to wait between 10 and 20 minutes at the bus stop ?

(b) Consider two populations $\pi_1$ and $\pi_2$ having density functions $P_1(x)$ and $P_2(x)$ respectively. Suppose a measurement $x_0$ is recorded on a new item yielding the density values $P_1(x_0) = 0.2$, $P_2(x_0) = 0.3$. Assign this item to population $\pi_1$ or $\pi_2$ given the following information. 9

The cost of misclassifying items as $\pi_2$ is 75 and misclassifying items as $\pi_1$ is 50. Further 30% of all items belong to $\pi_1$.

5. (a) Suppose that the lifetimes $X_1, X_2, \ldots$ of a component are i.i.d. with a uniform distribution on [0,10]. Let $0<T<10$ and suppose age replacement policy is to be employed. 6

   (i) Find $\mu^T$, the mean renewal time.

   (ii) Suppose that each replacement costs 2 units of money. An additional cost of 8 units is incurred if failure occurs. Out of $T=5$ and $T=6$, which one will result in lesser long run average cost per unit time ?

(b) Consider a grocery shop with two cashiers. The average time to prepare a bill and complete payment for a customer is 3 minutes for each cashier. Customers arrive at the cash counters at a rate of 30 per hour, and go to the first available cash counter on a first come first served basis. 9

   (i) Find $P_0$.

   (ii) Find the average number of customer in the waiting line.

   (iii) Find the average time a customer spends in the queue.

   (iv) Find the average time a customer spends in the system.

(v) Find the average number of customers in the system.

6. (a) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is given by 8

$$\Sigma = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

(b) Let $\{X_n : n \geq 1\}$ be an i.i.d. Sequence of inter-occurrence times with common p.d.f. $f(x)$ 7

given by $f(x) = \begin{cases} e^{-(x-2)} & \text{if } x > 2 \\ 0 & \text{otherwise} \end{cases}$

(i) Find the Laplace transform $\widetilde{F_t}$ of the distribution function F.

(ii) Find Laplace transform $\widetilde{M_t}$ of the renewal function $M_t$ for the corresponding renewal process.

7. (a) Find the stationary distribution of the Markov chain with transition probability matrix 5

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

(b) Find the sample correlation matrix for the data matrix 5

$X = \begin{bmatrix} 4 & 2 \\ 1 & 5 \\ 4 & 8 \end{bmatrix}$. Here 3 is the sample size.

(c) Suppose random variables X and Y have joint probability density function f given by **5**

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Check whether X and Y are independent.

**8.** State whether the following statements are true or false. Justify your answer with valid reasons. **10**

(a) For events A1,A2... and B with P(B)>0,

$$P\left( \bigcup_{i=1}^{\infty} Ai \mid B \right) = \sum_{i=1}^{\infty} P(Ai \mid B)$$

(b) If A = $(a_{ij})$ be the matrix of the quadratic form $ax_1^2 + bx_2^2 + cx_1x_2$, then $a_{12} = 2c$.

(c) If $N_t$ be a renewal process for which the inter occurance times $X_i$'s have finite mean, then

$$\lim_{t \to \infty} \frac{Mt}{t} > 0$$

(d) If $X_1, X_2 \ldots X_n$ be independent observations from a population with mean $\mu$ and covariance matrix then $\Sigma_{pxp}$, then

$n(\overline{X} - \mu)' S^{-1} (\overline{X} - \mu)$ is approximately $\chi_p$.

(e) In a Markov chain if $f_{ii} = 1$ for some state i and i communicates with j, then $f_{jj} = 1$.