MMT - 003

ASSIGNMENT BOOKLET

Masters in Mathematics with Applications to Computer Science

Algebra

(Valid from 1st January, 2024 to 31st December, 2024.)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi
(2024)

Dear Student,

Please read the section on assignments in the Programme Guide for elective Courses that we sent you after your enrolment. As you may know already from the programme guide, the continuous evaluation component has 30% weightage. This assignment is for the continuous evaluation component of the course.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully.

i) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :..........................

NAME :..........................

ADDRESS :..........................

....

....

COURSE CODE : ....

COURSE TITLE : ....

STUDY CENTRE : .... **DATE**..........................

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

ii) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

iii) Leave a 4 cm margin on the left, top and bottom of your answer sheet.

iv) Your answers should be precise.

v) While solving problems, clearly indicate which part of which question is being solved.

vi) This assignment is valid only up to December, 2024. If you fail in this assignment or fail to submit it by December, 2024, then you need to get the assignment for 2025 and submit it as per the instructions given in the Programme Guide.

We **strongly** suggest that you retain a copy of your answer sheets.

Wish you good luck.

Assignment

Course Code: MMT-003
Assignment Code:MMT-003/TMA/2024
Maximum Marks: 100

1. Which of the following statements are true and which are false? Give reasons for your answer. (10)
   (a) If a finite group G acts on a finite set S, then $G_{s_1} = G_{s_2}$ for all $s_1, s_2 \in X$.
   (b) There are exactly 8 elements of order 3 in S_4.
   (c) If $F = \mathbb{Q}(\sqrt[5]{2}, \sqrt[3]{5})$, then $[F : \mathbb{Q}] = 8$.
   (d) $\mathbf{F}_7\left(\sqrt{3}\right) = \mathbf{F}_7\left(\sqrt{5}\right)$.
   (e) For any $\alpha \in \mathbb{F}_{2^5}^*$, $\alpha \neq 1$, $\mathbb{F}_{2^5} = \mathbb{F}_2[\alpha]$.

2. (a) Consider the natural action of $GL_2(\mathbb{R})$ on $\mathbf{M}_2(\mathbb{R})$, the set of 2×2 real matrices, by left multiplication.
   (i) Under this action, if $\det(\mathbf{x}) \neq \mathbf{0}$, show that the stabiliser of $\mathbf{x} \in M_2(\mathbb{R})$ is $\{\mathbf{I}\}$, where $\mathbf{I}$ is the 2×2 identity matrix.
   (ii) Suppose that $\det(\mathbf{x}) = \mathbf{0}$ in the remaining parts of this exercise. We will show that the stabiliser of $\mathbf{x}$ is infinite. If $\mathbf{x} = \mathbf{0}$, the stabiliser of $\mathbf{x}$ is $GL_2(\mathbb{R})$. So suppose $\mathbf{x} \neq \mathbf{0}$. Let us write $\mathbf{x} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then, $\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} c \\ d \end{bmatrix}$ for non-zero $\lambda \in \mathbb{R}$. Why ?
   (iii) Let $\begin{bmatrix} a' \\ b' \end{bmatrix}$ be a vector that is not a scalar multiple of $\begin{bmatrix} a \\ b \end{bmatrix}$. Show that there is a matrix $\mathbf{b} = \begin{bmatrix} u & v \\ w & z \end{bmatrix}$ such that $\mathbf{b}\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0}$ and $\mathbf{b}\begin{bmatrix} a' \\ b' \end{bmatrix} = \alpha \begin{bmatrix} a' \\ b' \end{bmatrix}$.(**Hint:** Set up two sets of simultaneous equations in two unknowns and argue why they have a solution.)
   (iv) Check that $\mathbf{I} - \mathbf{b}$ is in the stabiliser of $\mathbf{x}$. Also, show that there are infinitely many choices of α for which $\mathbf{I} - \mathbf{b}$ is invertible. (5)

   (b) Let H be a finite group and, for some prime p, let P be a p-Sylow subgroup of H which is normal in H. Suppose H is normal in K, where K is a finite group. Then, show that P is normal in K. (3)
   (c) Find the elementary divisors and invariant factors of $\mathbb{Z}_8 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$. (2)

3. Describe the set of primes p for which $x^2 - 11$ splits into linear factors over $\mathbb{Z}_p$. (10)

4. (a) Determine, up to isomorphism, all the finite groups with exactly 2 conjugacy classes. (4)
   (b) Is there a finite group with class equation $1 + 1 + 2 + 2 + 2 + 2 + 2 + 2$? (3)
   (c) Compute the following:
   a) $\left(\frac{173}{211}\right)$ b) $\left(\frac{167}{239}\right)$. (3)

5. (a) Let $F(\alpha)$ be a finite extension F of odd degree(greater than 1). Show that $F\left(\alpha^2\right) = F(\alpha)$. (2)
   (b) Let $F \subset K$ and let $\alpha, \beta \in K$ be algebraic over F of degree m and n, respectively. Show that $[F(\alpha, \beta) : F] \leq mn$. What can you say about $[F(\alpha, \beta) : F]$ if m and n are coprime? (5)
   (c) Find $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}]$ where $\omega^3 = 1, \omega \neq 1$. (3)

6. (a) If $char(F) \neq 2$, show that a polynomial $ax^2 + bx + c$ is irreducible iff $b^2 - 4ac \notin \mathbb{F}^{*2}$ where $\mathbb{F}^{*2}$ is the group of squares in $\mathbb{F}^*$. (2)

(b) By looking at the factorisation of $x^9 - x \in \mathbb{F}_3[x]$ guess the number of irreducible polynomials of degree 2 over $\mathbb{F}_3$. Find all the irreducible polynomials of degree 2 over $\mathbb{F}_3$. (6)

(c) If $\mathbb{F}$ is a finite field show that there is always an irreducible polynomial of the form $x^3 - x + a$ where $a \in F$.(**Hint:** Show that $x \mapsto x^3 - x$ is not a surjective map.) (2)

7. (a) Suppose that $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $2n \times 2n$ matrix where A, B, C and D are $n \times n$ matrices. Show that M is symplectic if and only if the following conditions are satisfied:

$$A^t D - C^t B = \mathbf{I}$$
$$A^t C - C^t A = \mathbf{0}$$
$$B^t D - D^t B = \mathbf{0}$$

(**Hint:** Use block matrix multiplication.)
Also, check that the matrix $\begin{bmatrix} \mathbf{0} & -A \\ A & \mathbf{0} \end{bmatrix}$, where A is a $n \times n$ orthogonal matrix, is a symplectic matrix. (4)

(b) The aim of this exercise is to show that $SP_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

(c) Show that

(i) Show that a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$ is symplectic if and only if $ad - bc = 1$. (1)

(ii) Show that, to prove that $SP_2(\mathbb{R})$ acts transitively on $GL_2(\mathbb{R})$, it is enough to show that, for any vector $\begin{bmatrix} a \\ b \end{bmatrix} \neq \mathbf{0} \in \mathbb{R}^2$, there is a 2×2 symplectic matrix with $\begin{bmatrix} a \\ b \end{bmatrix}$ as the first column. (**Hint:** For any matrix A, what is $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?) (3)

(iii) Complete the proof by showing that, given any non-zero vector $\begin{bmatrix} a \\ b \end{bmatrix}$, there is always a non-zero vector $\begin{bmatrix} a' \\ b' \end{bmatrix}$ such that $\begin{bmatrix} a & a' \\ b & b' \end{bmatrix}$ is symplectic. (2)

8. In this exercise, we ask you to find the Sylow p-subgroups of the dihedral group

$$D_n = \langle x, y \,:\, x^n, y^2, yxyx \rangle, n \in \mathbb{N}, n \geq 2.$$

(a) Let p be an odd prime that divides n, $n = p^r l$, $p \nmid l$. Suppose $C = \langle x^l \rangle$. Show that C is the unique Sylow p-subgroup of D_n. (3)

(b) Prove the relation

$$y^i x^j y^k x^l = \begin{cases} y^i x^{j+l} & \text{if k is even} \\ y^{i+k} x^{l-j} & \text{if k is odd.} \end{cases}$$

Further, find all the elements of order 2 in D_n. (4)

(c) Find all the Sylow 2-subgroups of D_n when n is odd. Describe them in terms of x and y. (2)

(d) Suppose n is even, $n = 2^k m$, where $2 \nmid m$, $k \geq 2$. Let $N = \langle x^m \rangle$ and $H = \langle y \rangle$. Show that HN is a subgroup of D_n. What is its order? (3)

(e) Suppose n is as in the previous part. Find all the Sylow 2-supgroups of D_n. Describe them in terms of x and y. (3)

9. (a) Let $G = \left\langle a, b \,\middle|\, a^2, b^3, aba^{-1}b^{-1} \right\rangle$. Show that G is the cyclic group of order six. (5)

(b) Solve the following set of congruences: (5)

$$\begin{aligned} x &\equiv 2 \pmod{17} \\ 3x &\equiv 4 \pmod{19} \\ x &\equiv 7 \pmod{23} \end{aligned}$$

(c) Show that $\mathbb{Q}\left(\sqrt{-19}\right)$ is not a UFD by giving two different factorisations of 20. (5)

Question no.	Block 1	Block 2	Block 3	Block 4	Block 5
2 a)	5				
2 b)	3				
2 c)		2			
3 c)			10		
4 a)	4				
4 b)	3				
4 c)			3		
5 a)				2	
5 b)				5	
5 c)				3	
6 a)				2	
6 b)				6	
6 c)				2	
7 a)		4			
7 c)		1			
7 c)		3			
7 c)		2			
8 a)	3				
8 b)	4				
8 c)	2				
8 d)	3				
8 e)	3				
9 a)		5			
9 b)			5		
9 c)			5		
Total	30	17	23	20	0