MMT-006

ASSIGNMENT BOOKLET

M.Sc. (Mathematics with Applications in Computer Science)
FUNCTIONAL ANAYLSIS
(Valid from 1st January, 2023 to 31st December, 2023)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2023)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE: ………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2023. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by December, 2023, then you need to get the assignment for 2024, and submit it as per the instructions given in the programme guide.
8) **You cannot fill the Exam Form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment

Course Code: MMT-006
Assignment Code: MMT-006/TMA/2023
Maximum Marks: 100

1. State whether the following statements are **true** or **false**. Justify with a short proof or a counter example. (5×2=10)

   a) The space l^3 is a Hilbert space

   b) Any non zero bounded linear functional on a Banach space is an open map.

   c) Every bounded linear map on a complex Banach space has an eigen value.

   d) The image of a Cauchy sequence under a bounded linear map is also a Couchy sequence.

   e) If A is a bounded linear operator on a Hilbert space such that $AA^* = I$, then $A^*A = I$.

2. a) Characterise all bounded linear functionals on a Hilbert space. (5)

   b) Show that the map $T: \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3)$ is an open map. (2)

   c) Check whether a finite dimensional normed linear space is reflexive? Justify your answer. (3)

3. a) Show how a real linear functional u on a complex linear normed space gives rise to a complex linear functional f. What is the relation between the boundedness of u and that of f? (5)

   b) In a Hilbert space. Prove that $x_n \to x$ provided $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$. (2)

   c) Are Hahn-Banach extensions always unique? Justify. (3)

4. a) State the principle of uniform boundedness. Use it to show that a set E in a normed space X is bounded if $f(E)$ is bounded in K for every $f \in X'$. (5)

   b) If H is a Hilbert space and SCH, show that $S^{\perp} = S^{\perp\perp\perp}$. When S is the same as $S^{\perp\perp}$? Justify. (5)

5 a) Show that Q defined on $(C[0, 1], \|\cdot\|_{\infty})$ by $Q(x) = \int_0^1 t\, x(t) dt$ is a bounded linear functional. Calculate $\|Q\|$. (3)

b) Let A be an operator on a Hilbert space H. Show if $\|Ax\| = \|A^* x\|$ for every $x \in H$, then A is normal. Is it converse true? Justify. (4)

c) Let $\{u_n\}$ be the sequence in l^2 with 1 in the n^{th} place and zeroes else where prove that the set $\{u_n\}$ is an orthonormal basis for l^2. (3)

6 a) Let $X = C[0, 1]$ with Sup norm defined by $\|f\| = \underset{x \in [0, 1]}{\operatorname{Sup}}\{|f(x)|\}$.

Let T be a linear map defined on X by $T(f) = f\left(\frac{1}{2}\right)$.

Show that T is a bounded linear map such that $\|T\| = 1$. (4)

b) Define Eigen Spectrum of a bounded linear operator on a Banach space. Show that the eigen spectrum of the operator T on l^2 given by $T(\alpha_1, \alpha_2 \ldots) = (0, \alpha_1, \alpha_2 \ldots)$ is empty. (3)

c) Let $\|\cdot\|$ be a norm on a linear space X. if $x, y \in X$ and $\|x + y\| = \|x\| + \|y\|$, then show that $\|sx + ty\| = s\|x\| + t\|y\|$ for all $s \geq 0, t \geq 0$. (3)

7. a) Let X be an inner product space with the inner product given by $<, >$. For $x \in X$, define the function $\|\cdot\| : X \to K$ given by $\|x\| = < x, -x > 1/2$, the non negative square root of $< x, x >$. Show that $\|\cdot\| : X \to K$ defines a norm on X and $|< (x, y) >| \leq \|x\| \, \|y\|$ for all $x, y \in X$. Also show that for all $x, y \in X$,

$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ (5)

b) Let X be a vector space. Let $\|\cdot\|^1$ and $\|\cdot\|^2$ be two norms on X. When are these norms said to be equivalent? Justify your answer.

Let $X = \mathbb{R}^3$. For $x = (x_1, x_2, x_3)$.

Let $\|x\|^1 = |x_1| + |x_2| + |x_3|$

$\|x\|^2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$

Show that $\|\cdot\|^1$ and $\|\cdot\|^1$ are equivalent. (5)

8. a) Let $X = L^2[0, 2\pi]$ and $u_n(t) = \dfrac{e^{int}}{\sqrt{2\pi}}$, $t \in \{-\pi, \pi\}$, $n \in Z$. Show that the set $E = \{u_1, u_2 \ldots\}$ is an orthonormal set in X. (3)

b) Let H be a Hilbert space. For any subset A of H, define $A^\perp$. If $A \subseteq B \subseteq H$, then show that:

i) $B^\perp \subseteq A^\perp$

ii) $A \subseteq A^{\perp\perp}$

State conditions on A so that $A^{\perp\perp} = A$. (5)

c) Let $X = C'[0, 1]$ and $Y = C[0, 11]$ and let $T : X \to Y$ be the linear operator from X to Y given by $T(f) = f'$, the derivative of f on $[0, 1]$. Show that T is not continuous. (2)

9. a) Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is continuous and open. Will F always be closed? Will F be always surjective? Give reasons for your answer. (3)

b) Check whether the identity map on an infinite dimensional space is compact. (3)

c) Define $A : \mathbb{C}^3 \to \mathbb{C}^3$ by $A(z_1, z_2, z_3) = (iz_1, e^{2i}z_2, z_3)$.

Check whether A is i) self adjoint, ii) unitary. (4)

10. a) Let $A : X_0 \subseteq X \to Y$ be a closed operator where X and Y are Banach spaces.

Define $\|x\|_A = \|x\| + \|Ax\|$, $x \in X_0$.

Then show that the norm $\|\cdot\|_A$ is complete. (4)

b) Find a bounded linear functional f on ℓ^3 such that $f(e_3) = 3$ and $\|f\| = 3$. (2)

c) Prove that $l^1 \subset l^2$. If: $T : (l^2, \|\cdot\|_2) \to (l^1, \|\cdot\|_2)$ is a compact operator, show that:

$T : (l^2, \|\cdot\|_2) \to (l^2, \|\cdot\|_2)$ is also compact. (4)