MMT-006

ASSIGNMENT BOOKLET

M.Sc. (Mathematics with Applications in Computer Science)

FUNCTIONAL ANALYSIS

(Valid from 1st July, 2021 to 30th June, 2022)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2021-22)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME:……………………………………………

ADDRESS:……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE: …………………………….

ASSIGNMENT NO. ………………………….…

STUDY CENTRE: ………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto June, 2022. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by June, 2022, then you need to get the assignment for the year 2022-23 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment

Course Code: MMT-006
Assignment Code: MMT-006/TMA/2021-22
Maximum Marks: 100

1. Are the following statements True or False? Justify your answers with the help of a short proof or a counter example. (5×2=10)

   a) $L^1[a,b]$ is a reflective space.

   b) Every Banach space is also a Hilbert space.

   c) There exists an infinite dimensional Banach space X such that every proper subspace of X is complete.

   d) If the dual X' of a normed linear space X is finite dimensional, then X is finite dimensional.

   e) If $T: X \to Y$ is continuous, linear and open, then T is surjective.

2. a) State the open mapping theorem and deduce the closed graph theorem from it. (3)

   b) Let $\|.\|_1, \|.\|_2$ be norms on a linear space X and let $\varepsilon > 0, \delta > 0$ be fixed. Find conditions on the norms so that $B_1(0,\varepsilon) \subset B_2(0,\delta)$ holds. (3)

   c) Check whether the following sets are subspaces of $l^p, 1 \le p \le \infty$. (4)

   i) c_{oo} ii) c_o

3. a) For $x = (2,3) \in \mathbf{R}^2$, find $\| x \|_p$ when $p = 1, 3, 4, \infty$. (2)

   b) Consider $\mathbb{R}^2$ with $\|\cdot\|_2$. Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$ and $x = (-1,1)$. Find $d(x, M)$. (2)

   c) Show that a sequence $\{x_n + Y\}$ converges to $x + Y$ in X/Y if and only if there is a sequence $\{y_n\}$ in Y such that $\{(x_n + y_n)\}$ converges to x in X. (4)

   d) Let $X = \mathbf{R}^2$, $Y = \mathbf{R}^2$, each having norm $\|.\|_2$. Find the norm of the element $(1,2)$ in the product space $X \times Y$. (2)

4. a) Calculate the norm of the linear functional $f \mapsto \int_0^1 t f(t)\, dt$ on each of the spaces $(C[0,1], \|.\|_\infty)$ and $(L'[0,1], \|.\|_1)$. (4)

b) Let M be a proper closed subspace of a normed linear space X, $x_0 \notin M$ and $d = d(x_0, M)$. Proved that there is a bounded linear functional f_0 on X such that $\|f_0\| = \frac{1}{d}, f_0(x_0) = 1$ and $f_0(M) = 0$. (3)

c) If X is an infinite dimensional space, show that the set $\{x \in X : \| x \| = 1\}$ is not compact. (3)

5. a) Illustrate projection theorem on $L^2[0,1]$ with the subspace $M = \{f \in L^2[0,1] : f = 0 \text{ a.e. on } [0,1]\}$. (3)

b) For an orthonormal sequence $\{u_n\}$ in a Hilbert space H, prove that the following conditions are equivalent.

i) $x \perp u_n$ for all n implies $x = 0$,

ii) $x = \sum < x, u_n > u_n$ for all $x \in H$. (3)

c) Let X and Y be a normed linear spaces and T is a linear operator from X to Y. Then which of the following holds? Justify your answer.

i) If T is closed, then T is bounded

ii) If T is bounded, then T is closed. (4)

6. a) Given below is an important theorem along with a proof. In the proof certain steps are followed by a question. You should give answers to those questions.

Theorem: Show that a Banach space cannot have a countably infinite basis.

Proof: Suppose that X is an infinite dimensional Banach space having a countable basis $\{x_1, x_2, ...\}$. Let

$$Y_n = \text{span}\{x_1, ..., x_n\}, \quad n = 1, 2, ...$$

and V_n be the complement of Y_n in X. Then,

$$\bigcup_{n=1}^{\infty} Y_n = X, \quad \bigcap_{n=1}^{\infty} V_n = \phi. \qquad \dots (1)$$

Question: *Why the equalities in (1) hold?*

Hence, $\cap V_n$ cannot be dense in X. Since Y_n is finite dimensional, it is closed. Therefore, each V_n is open in X. As X is a complete metric space, Baire's theorem implies that some V_n must be nondense in X, that is, $\overline{V_N} \neq X$ for some N.

Question: *Why $\cap V_n$ is closed?*

Let $a \in X$ and $a \notin \overline{V}_N$. Then there is $r > 0$ such that the open ball $U(a, r)$ is disjoint from V_N, that is, $U(a, r) \subset Y_N$. Let x be any element in X. There is $\delta > 0$ such that $\delta\|x - a\| < r$.

Question: *Why does such a δ exist?*

Then $y = a + \delta(x - a) \in U(a, r) \subset Y_N$.

Since

$x = \frac{y-a}{\delta} + a = \frac{1}{\delta}y + \left(1 - \frac{1}{\delta}\right)a$, and $a \in Y_N$, we see that $x \in Y_N$. This shows that $X = Y_N$. So X is finite dimensional. This contradiction proves the result.

Question: *Why there is a contradiction?* ■ (8)

b) Use the theorem given in 6(a) to show that the space c_{00} is not a Banach space. (2)

7. a) Give an example of the following with proper justification.

i) An unbounded linear operator.

ii) An element of c_0 which does not belong to $l^p, 1 \le p < \infty$.

iii) A bounded linear operator with no eigenvalue. (6)

b) Let X be a normed linear space and $E \subset X$. Show that E is bounded in X if and only if $f(E)$ is bounded in $\mathbf{K}$ for every $f \in X'$. [State the theorem, lemmas and other results used in proving this. (4)

8. a) Prove that a Banach space X is reflexive if and only if its dual X' is reflexive. (6)

b) When is a bounded linear operator A on a Hilbert space H is called a Hilbert-Schmidt operator? Give an example of a Hilbert-Schmidt operator on $H = l^2$. Justify your choice of example. Find the adjoint of this operator. Is it Hilbert-Schmidt? (4)

9. a) Prove the following theorem by stating all the theorems and lemmas used:

Let A be a compact self-adjoint operator on X and let $\{\lambda_i : i \in \Delta\}$ be the set of all nonzero eigenvalues of A. For each $i \in \Delta$, let $\{u_1^{(i)}, ..., u_{m_i}^{(i)}\}$ be an orthonormal basis of $N(A - \lambda_i I)$. Then

$$Ax = \sum_{i \in \Delta} \sum_{j=1}^{m_i} \lambda_i \langle x, u_j^{(i)} \rangle u_j^{(i)} \quad \forall x \in X$$

and $\bigcup_{i\in\Delta}\{u_1^{(i)},...,u_{m_i}^{(i)}\}$ is an orthonormal basis of $N(A)^{\perp}$. In particular,

$$A = \sum_{i\in\Delta} \lambda_i P_i \text{ , where } P_i \text{ is the orthogonal projection onto } N(A-\lambda_i I).$$

The proof should include the answers to the following question also:

i) Why $\bigcup_{i\in\Delta}\{u_1^{(i)},....,u_{m_i}^{(i)}\}$ is an orthonormal basis?

ii) Why E_i for each i, is an orthogonal projection onto $N(A-\lambda_i I)$? (6)

b) Let $X = C_0$ with $\|.\|_\infty$. For every $y \in l'$, let $f_y : X \to K(\mathbf{R} \text{ or } \mathbf{C})$ be defined by

$$f_y(x) = \sum_{j=1}^{\infty} x(j)\, y(j), \ x \in X \ .$$

Show that $f_y \in X'$. (4)

10. a) Carefully read the proof of the Riesz representation theorem for Hilbert spaces and find out where the completeness of the space is used in the proof. Is it possible to have a continuous linear functional f on an incomplete inner product space so that the conclusion of Riesz representation theorem holds? Justify your answer. (4)

b) Let $X_1,...,X_n$ be normed linear spaces with norms $\|.\|_1,....,\|.\|_n$, respectively, and let $X = X_1 \times ... \times X_n$ be the product linear space. For $x = (x_1,..., x_n) \in X$ and $j \in \{1,...,n\}$, let $\pi_j(x)$ be the element in X with its $j-$th component as x_j and all other components as zeroes, i.e.

$$\pi_j(x) = (y_1,..., y_n), \quad y_i = \begin{cases} x_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that the norm given by

$$\|(x_1,...,x_n)\|_1 = \sum_{j=1}^{n} \|x_j\|$$

defines a norm on the product space. (4)

c) If f is a linear functional on a normed linear space whose zero space $Z(f)$ is closed, then prove that f is continuous. (2)