MMT-005

ASSIGNMENT BOOKLET

(Valid from 1st January, 2022to 31st December, 2022)

M.Sc. (Mathematics with Applications in Computer Science)

COMPLEX ANALYSIS

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2022)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: ………………………….…

STUDY CENTRE: …………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2022. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by December, 2022, then you need to get the assignment for the year 2022 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the Exam Form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment

Course Code: MMT-005
Assignment Code: MMT-005/TMA/2022
Maximum Marks: 100

1. Determine whether each of the following statement is true or false. Justify your answer with a short proof or a counter example.

   i) If $z = a + ib$, where a and b are integers, then $|1 + z + z^2 + \cdots + z^n| \geq |z|^n$ if $a > 0$.

   ii) If $f(z)$ and $\overline{f(z)}$ are analytic functions in a domain, then f is necessarily a constant.

   iii) A real-valued function $u(x, y)$ is harmonic in D iff $u(x, -y)$ is harmonic in D.

   iv) $\lim\limits_{n \to \infty} (n!)^{1/n} = \infty$.

   v) The inequality $|e^a - e^b| \leq |a - b|$ holds for $a, b \in D = \{w : \operatorname{Re} w \leq 0\}$.

   vi) If $f(z) = \sum\limits_{n=0}^{\infty} a_n (z-a)^n$ has the property that $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ converges, then f is necessarily an entire function.

   vii) If a power series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$ and if $b_n \in \mathbb{C}$ is such that $|b_n| < n^2 |a_n|$ for all $n \geq 0$, then $\sum\limits_{n=0}^{\infty} b_n z^n$ converges for $|z| < 1$.

   viii) If f is entire and $f(z) = f(-z)$ for all z, then there exists an entire function g such that $f(z) = g(z^2)$ for all $z \in \mathbb{C}$.

   ix) A mobius transformation which maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto itself and fixing $0, \infty$ and no other points, must be of the form $Tz = \alpha z$ for some $\alpha > 0$ and $\alpha \neq 1$.

   x) If f is entire and $\operatorname{Re} f(z)$ is bounded as $|z| \to \infty$, then f is constant. (20)

2. a) If $f = u + iv$ is entire such that $u_x + v_y = 0$ in $\mathbb{C}$ then show that f has the form $f(z) = az + b$ where a, **b** are constants with $\operatorname{Re} a = 0$. (5)

   b) Consider $f(z) = z^2 - z$ and the closed circular region $R = \{z : |z| \leq 1\}$. Find points in R where $|f(z)|$ has its maximum and minimum values. (3)

   c) Find the points where the function $f(z) = \dfrac{Log(z+4)}{z^2 + i}$ is not analytic. (2)

3. a) Evaluate the following integrals:

   i) $I = \int\limits_0^{2\pi} f(e^{i\theta}) \cos^2(\theta/2)\, d\theta$.

ii) $I = \int_0^{2\pi} f(e^{i\theta}) \sin^2 \theta/2 \, d\theta$. (6)

b) Find the image of the circle $|z| = r \, (r \neq 1)$ under the mapping $w = f(z) = \dfrac{z-i}{z+i}$. What happens when $r = 1$? (4)

4. a) If $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \, (n \geq 1)$, then show that there exists a real $R > 0$ such that $2^{-1} |z|^n \leq |p(z)| \leq 2|z|^n$ for $|z| \geq R$. (5)

b) Find all solutions to the equation $\sin z = 5$. (5)

5. a) Find the constant c such that $f(z) = \dfrac{1}{z^n + z^{n-1} + \cdots + z^2 + z^{-n}} + \dfrac{c}{z-1}$ can be extended to be analytic at $z = 1$, when $n \in \mathbb{N}$ is fixed. (3)

b) Find all the singularities of the function $f(z) = \exp\left(\dfrac{z}{\sin z}\right)$. (3)

c) Evaluate $\oint_C \dfrac{dz}{z^2+1}$ where c is the circle $|z| = 4$. (4)

6. a) Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$. (4)

b) Evaluate $\int_C \dfrac{z^3+3}{z(z-i)^2} \, dz$, where c is the eight like figure shown in Fig. 1. (3)

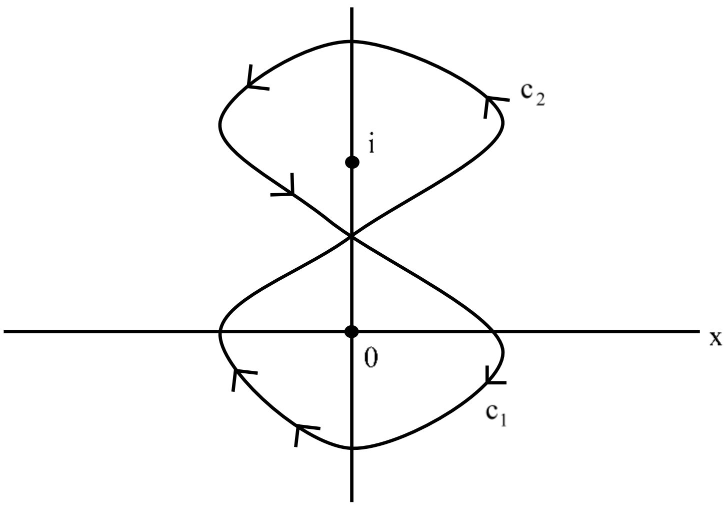

Fig. 1

c) Find the radius of convergence of the following series.

i) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} (z-1-i)^k$ ii) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$ (3)

7. a) Expand $f(z)=\dfrac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for

i) $0<|z-1|<2$ and ii) $0<|z-3|<2$. (4)

b) Find the zeros and singularities of the function $f(z)=\dfrac{z}{4\cos^2 z-1}$ in $|z|\le 1$. Also find the residue at the poles. (3)

c) Prove that the linear fractional transformation $\phi(z)=\dfrac{2z-1}{2-z}$ maps the circle $c:|z|=1$ into itself. Also prove that $f(z)$ is conformal in $\overline{D}=\{z:|z|\le 1\}$. (3)

8. a) Find the image of the semi-infinite strip $x>0,\ 0<y<1$ when $w=i/z$. Sketch the strip and its image. (5)

b) Show that there is only one linear fractional transformation that maps three given distinct points z_1, z_2 and z_3 in the extended z plane onto three specified distinct points w_1, w_2 and w_3 in the extended w plane. (5)

9. Evaluate the following integrals

a) $\displaystyle\int_0^\infty \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx$. (5)

b) $\displaystyle\int_{-\infty}^\infty \frac{\sin^2 2x}{1+x^2}\,dx$. (5)