MMT-004

ASSIGNMENT BOOKLET

(Valid from 1st January, 2021 to 31st December, 2021)

M.Sc. (Mathematics with Applications in Computer Science)

REAL ANALYSIS (MMT-004)

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068
(2021)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective courses that we sent you after your enrolment. A weightage of 20 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE: …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO. ………………………….…

STUDY CENTRE: …………………………..….. DATE: ……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved..
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto December, 2021. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by December, 2021, then you need to get the assignment for the year 2022 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment (MMT – 004)

Course Code: MMT-004
Assignment Code: MMT-004/TMA/2021
Maximum Marks: 100

1. State whether the following statements are true or false. Give reasons for your answers. (10)

   a) If (X, d_1) is a discrete metric space and (Y, d_2) is any metric space, then any function $f : X \to Y$ is continuous.

   b) Continuous image of a Cauchy sequence in a metric space is a Cauchy sequence.

   c) A discrete metric space has no dense subset.

   d) If $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

   $$f(x,y) = \begin{cases} |x| + |y|, & \text{if } (x,y) \neq (0,0) \\ 0 \quad , & \text{elsewhere} \end{cases}$$

   then the directional derivative of f along $(0,1)$ does not exist at $(0,0)$.

   e) The outer measure of a subset of $\mathbf{R}$ is always finite.

2. a) Use Urysohn's lemma to prove the following result:

   Let E and F be non-empty disjoint closed subsets of a metric space (X, d). Then, there exist open sets $A \supset E$ and $B \supset F$ such that $A \cap B = \phi$. (4)

   b) Find the interior, closure and boundary of the set $A = \{(x,0) \in \mathbf{R}^2 : 0 < x \leq 1\}$ as a subset of $\mathbf{R}^2$ with standard metric. (3)

   c) Find the components of $\mathbf{Q}$ under the standard metric. (3)

3. a) Verify whether the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

   $$f(x,y,z) = \begin{cases} \left(y, x, z^2 \sin\frac{1}{z} \right) & \text{if } \ (x,y,z) \neq (0,0,0) \\ (0,0,0) & \text{if } \ (x,y,z) = (0,0,0) \end{cases}$$

   is continuously differentiable or not at $(0,0,0)$. (3)

   b) Verfiy the Implicit Function Theorem for the function $f : \mathbf{R}^5 \to \mathbf{R}^2$ defined by $f(x_1, x_2, y_1, y_2, y_3) = (2e^{x1} + x_2 y_1 - 4y_2 + 3, x_2 \cos x_1 - 6x_1 + 2y_1 - y_3)$ at $(0,1,3,2,7)$. (4)

   c) Obtain the Taylor's series expansion for $f(x,y) = e^{xy}$ at $(1,1)$. (3)

4. a) Find the critical points of the function, f, given by $f(x,y,z) = 2x^2 - 2y^2 + 4yz - 3z^2 - x^4 + 5$ and check whether they are extreme points. (4)

b) Use Lagrange's Multipliers Method to find the point on the plane $2x - 2y + z = 4$ that is closest to the origin. (4)

c) Which of the following sets are totally bounded? Give reasons for your answer.

i) $\mathbf{N}$ (with discrete metric)
ii) $(0,2) \cup [5,10)$ (2)

5. a) Find the Fourier series for (5)

$$f(x) = \begin{cases} -\pi/4 & \text{for } -\pi < x < 0 \\ \pi/4 & \text{for } 0 < x < \pi \end{cases}$$

b) Prove that if $g(x) = \overline{f(-x)}$, then $\hat{g}(\omega) = \overline{\hat{f}(\omega)}$. (2)

c) Find the outer measure of the following sets:

i) $A = \{x \in \mathbf{R} \mid \cos 2x = 1\}$
ii) $B = \{x \in \mathbf{R} \mid |x - 5| \le 7\}$ (3)

6. a) Check if the conditions and conclusions of Fatou's lemma is satisfied for a sequence of function $\{f_n\}$ where $f_n(x) = \chi_{[n,n+3]}$. (4)

b) Let f be a measurable defined over an interval $[a,b]$. Show that f is Lebesgue integrable over $[a,b]$ if and only if $|f|$ is Lebesgue integrable over $[a,b]$. If in particular, f is Lebesgue integrable over $[a,b]$, then show that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. (4)

c) Check whether a finite set in a metric space is nowhere dense. (2)

7. a) If A and B are open subsets of $(\mathbf{R}, d)$ where d is the Euclidean metric on $\mathbf{R}$, then show that $A \times B$ is an open subset of $(\mathbf{R}^2, d)$ where d_2 is the Euclidean metric on $\mathbf{R}^2$. (3)

b) Let (X,d) be a metric space. If F is a closed subset of X and K is a compact subset of X, then show that $F \cap K$ is compact. (3)

c) Let (X,d) and (Y,d') be metric spaces. If $f : X \to Y$ and $g : X \to Y$ are two continuous function, then show that $F = \{x \in X : f(x) = g(x)\}$ is a closed subset of X. Hence deduce that if $f(x) = g(x) \forall x \in D$, where D is a dense subset of X, then $f = g$. (4)

8. a) Show that a metric space (X,d) is connected if and only if for each $a, b \in X$, there is a connected subset E of X such that $a, b \in E$. (5)

b) Let $\{x_n\}$ and $\{y_n\}$ be cauchy sequences in a metric space X. Show that the sequence $\{d(x_n, y_n)\}$ converges in $\mathbf{R}$. (3)

c) Find the Fourier transform of the function $f(x) = e^{(x-2)}$. (2)

9. a) Verify the inverse function theorem for the function $f : E \to \mathbf{R^2}$ defined by $f(x, y) = (x^2 - y^2, 2xy)$ where $E = \{(x, y) \mid x \neq 0\}$ is a subset of $\mathbf{R^2}$. (4)

b) Verify Monotone convergence theorem for $\{f_n\}$ where $f_n \to \mathbf{R} \to \mathbf{R}$ is defined by

$$f_n(x) = \begin{cases} 1, & x \in \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right]. \\ 0, & \text{elsewhere} \end{cases}$$ (4)

c) Let X and Y be metric spaces and f is continuous function from X to Y. If E is non-empty compact subset of X, then show that $f(E)$ is compact in Y. (2)

10. a) Lebesgue integration has wider scope than Riemann integration. Justify this statement with two examples. (4)

b) Give an example each of the following. Justify your choice of examples. (6)

i) A non-trivial set in Euclidean space $\mathbf{R}^2$ which is not path connected.
ii) A vector-valued function $f : \mathbf{R}^4 \to \mathbf{R}^3$ for which the second order derivative exists.
iii) A Lebesgue integrable function which is not differentiable.