MMT-008

ASSIGNMENT BOOKLET

(Valid from 1st July, 2021 to 30th June, 2022)

M.Sc. (Mathematics with Applications in Computer Science)

PROBABILITY AND STATISTICS (MMT-008)

School of Sciences
Indira Gandhi National Open University
Maidan Garhi, New Delhi-110068

(2021-22)

Dear Student,

Please read the section on assignments and evaluation in the Programme Guide for Elective courses that we sent you after your enrolment. A weightage of 20 per cent, as you are aware, has been assigned for continuous evaluation of this course, **which would consist of one tutor-marked assignment**. The assignment is in this booklet.

Instructions for Formatting Your Assignments

Before attempting the assignment please read the following instructions carefully.

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO :……………………………………………
NAME :……………………………………………
ADDRESS :……………………………………………
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COURSE CODE: …………………………….
COURSE TITLE : …………………………….
ASSIGNMENT NO. …………………………….
STUDY CENTRE: ……………………………..   DATE: ……………………………………………

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.
3) Leave 4 cm margin on the left, top and bottom of your answer sheet.
4) Your answers should be precise.
5) While solving problems, clearly indicate which part of which question is being solved.
6) This assignment is to be submitted to the Programme Centre as per the schedule made by the programme centre. Answer sheets received after the due date shall not be accepted.

   We strongly suggest that you retain a copy of your answer sheets.
7) This assignment is valid only upto June, 2022. For submission schedule please read the section on assignments in the programme guide. If you have failed in this assignment or fail to submit it by June, 2022, then you need to get the assignment for the session 2022-23 and submit it as per the instructions given in the programme guide.
8) **You cannot fill the exam form for this course** till you have submitted this assignment. So solve it and **submit it to your study centre at the earliest.**

We wish you good luck.

Assignment (MMT – 008)

Course Code: MMT-008

Assignment Code: MMT-008/TMA/2021-22

Maximum Marks: 100

1. State whether the following statements are True or False. Justify your answer with a short proof or a counter example: (10)

   a) If P is a transition matrix of a Markov Chain, then all the rows of $\lim_{n\to\infty} P^n$ are identical.

   b) Every non-negative definite matrix is a variance-covariance matrix.

   c) The multiple correlation coefficient R can lie between – 1 and 0.

   d) Posterior probabilities obtained from Baye's theorem are larger than respective prior probabilities.

   e) If X_1, X_2, X_3 are iid from $N_2(\mu, \Sigma)$, then $\frac{X_1 + X_2 + X_3}{3}$ follows $N_2\left(\mu, \frac{1}{3}\Sigma\right)$.

2. a) Let the random vector $X' = [X_1\, X_2\, X_3]$ has mean vector

   $$= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \text{ and Var-cov matrix } = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix}. \text{ Fit the equation } Y = b_0 + b_1 X_1 + b_2 X_2.$$

   Also obtain the multiple correlation coefficient between X_3 and $[X_1, X_2]$. (5)

   b) Let N_s be a Poisson process with parameter $\lambda > 0$. Fix $s > 0$ and let the renewal function be given by $M_t = N_{(t+s)} - N_s$. Show that the conditional distribution of M_t, given $N_s = 10$, is Poisson. (5)

3. a) A barber shop has two barbers. The customers arrive at a rate of 5per hour in a Poisson fashion, and the service time of each barber takes an average of 15 minutes according to exponential distribution. The shop has 4 chairs for waiting customers. When a customer arrives in the shop and does not find an empty chair, she leaves the shop. What is the expected number of customers in the shop? What is the probability that a customer will leave the shop finding no empty chair to wait? (5)

b) In a branching process, the offspring distribution (p_k) is given below:

$p_k = pq^k,\ q = 1 - p, k = 0,1,2,...$

What will be the probability of extinction in this branching process? (5)

4. a) Let X be $N_3(\mu, \Sigma)$ with $\sum = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Examine the independence of the following:

i) X_1 and X_2;

ii) (X_1, X_2) and X_3;

iii) $X_1 + X_2$ and X_3. (6)

b) Let $\{N_n, n = 0,1,2,....\}$ be a renewal process with sequence of renewal periods $\{X_i\}$. Each X_i follows the binomial distribution with $P[X_i = 0] = 0.6$ and $P[X_i = 1] = 0.4$. Find the distribution of N_n. (4)

5. a) To fit linear are regression on dependent variable Y and independent variables X_1 and X_2 we have the following information:

$E(X_1) = 3, E(X_2) = 2, \text{Var}(X_1) = 2, \text{Var}(X_2) = 1, \text{Cov}(X_1, X_2) = 1, \text{Cov}(X_1, Y) = 3,$
$\text{Cov}(X_2, Y) = 1, V(Y) = 9.$ Find the multiple correlation coefficient R. (4)

b) Let the joint probability density function of two continuous random variables X and Y be

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

i) Find the marginal p.d.f of X and Y.

ii) Test independence of X and Y.

iii) Compute $P[0 < x < 0.4 \mid 0.3 < y < 0.8]$.

iv) Find $V(Y \mid X = x)$. (6)

6. a) Let $X \sim N_3(\mu, \Sigma)$ where $\mu = [2\ 1\ 3]'$ and $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find the distribution of

$\begin{bmatrix} X_1 - X_2 + X_3 \\ X_1 + X_2 + 2X_3 \end{bmatrix}$. (6)

b) A Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ has initial distribution $u_0 = [0.1, 0.3, 0.6]'$ and transition matrix $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$ having states (1, 2, 3). Obtain the following:

i) $P[X_2 = 3]$

ii) $P[X_1 = 2, X_2 = 3]$

iii) $P[X_0 = 1, X_1 = 3, X_2 = 2]$ (4)

7. a) Determine the principle components, Y_1 and Y_2, for the covariance matrix $\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$.

Also, calculate the proportion of total population variance for the first principal component. (5)

b) The mean Poisson rate of arrival of planes at an airport during peak hours is 20 per hour. 60 planes per hour can land at the airport in good weather and 30 planes per hour in bad weather in Poisson fashion. Find the following during peak hours:

i) The average number of planes flying over the field in good weather;

ii) The average number of planes flying over the field in bad weather;

iii) The average number of planes flying over the field and landing in good weather;

iv) The average number of planes flying over the field and landing in bad weather;

v) The average landing time in good weather and bad weather. (5)

8. a) An equal number of balls are kept in three boxes B_1, B_2 and B_3. The boxes B_1, B_2 and B_3 contain respectively 3%, 5% and 2% defective balls. One of the boxes is selected at random and a ball is drawn randomly. If the ball is found to be defective, what is the probability that it has come from B_2? (5)

b) Let $Z = [Z^{(1)}, Z^{(2)}]$ and

$$\text{Cov}(Z) = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{21} & \rho_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.4 & 0.5 & 0.6 \\ 0.4 & 1 & 0.3 & 0.4 \\ \hline 0.5 & 0.3 & 1 & 0.2 \\ 0.6 & 0.4 & 0.2 & 1 \end{array}\right]$$

Compute the correlation between the first pair of canonical variates and their component variables. (5)

9. a) From the samples of sizes 80 and 100 from two populations, the following summary statistics were obtained:

$$X_1 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, X_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

where X_1, X_2 are the means and S_1, S_2 are the standard deviations of two populations. Test for the equality of the population means at 5% level of significance. Assume $\sum_1 = \sum_2$. (5)

b) Describe birth and death processes with the parameter λ. If $\lambda_k = \lambda$ and $\mu_k = k\mu, k \geq 0, \lambda, \mu > 0,$ then show that the stationary distribution of these process always exists. Obtain the steady state distribution. (5)

10. a) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2 where $X_1 \sim N_2(\mu^{(1)}, \Sigma)$ and $X_2 \sim N_2(\mu^{(2)}, \Sigma), \mu^{(1)} = [1\ 2]', \mu^{(2)} = [-1\ 0]'$ and $\Sigma = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

Considering equal cost and equal prior probabilities, classify the observation $[-1\ 1]'$ in one of the two populations. (6)

b) Suppose interoccurance times $\{X_n : n \geq 1\}$ are uniformly distributed on $[0\ 1]$:

i) Find $\overline{M}_t$, the Laplace transform of the renewal function M_t.

ii) Find $\lim_{t\to\infty} M_t / t$. (4)