MTE-06

ASSIGNMENT BOOKLET

Bachelor's Degree Programme

ABSTRACT ALGEBRA

(Valid from 1st January, 2022 to 31st December, 2022)

It is compulsory to submit the assignment before filling in the exam form.

School of Sciences
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Maidan Garhi, New Delhi-110068

(2022)

Dear Student,

Please read the section on assignments in the Programme Guide for Elective Courses that we sent you after your enrolment. A weightage of 30 per cent, as you are aware, has been earmarked for continuous evaluation, **which would consist of one tutor-marked assignment** for this course. The assignment is in this booklet.

Instructions for Formating Your Assignments

Before attempting the assignment please read the following instructions carefully:

1) On top of the first page of your answer sheet, please write the details exactly in the following format:

ROLL NO.:……………………………………………

NAME :……………………………………………

ADDRESS :……………………………………………

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COURSE CODE : …………………………….

COURSE TITLE : …………………………….

ASSIGNMENT NO.: …………………………..…

STUDY CENTRE : ………………………..….. DATE :……………………….………………...

PLEASE FOLLOW THE ABOVE FORMAT STRICTLY TO FACILITATE EVALUATION AND TO AVOID DELAY.

2) Use only foolscap size writing paper (but not of very thin variety) for writing your answers.

3) Leave 4 cm margin on the left, top and bottom of your answer sheet.

4) Your answers should be precise.

5) While solving problems, clearly indicate which part of which question is being solved.

6) **This assignment is valid only upto December, 2022. If you have failed in this assignment or fail to submit it by the last date, then you need to get the assignment for the next cycle and submit it as per the instructions given in that assignment.**

7) It is compulsory to submit the assignment before filling in the exam form.

We strongly suggest that you retain a copy of your answer sheets.

We wish you good luck.

Assignment

Course Code: MTE-06
Assignment Code: MTE-06/TMA/2022
Maximum Marks: 100

1. Which of the following statements are true? Justify your answers. (This means that if you think a statement is false, give a short proof or an example that shows it is false. If it is true, give a short proof for saying so.) (20)

   i) $\phi(n) = n-1 \ \forall n \in \mathbb{N}$, where ϕ is the Euler-phi function.

   ii) If G_1 and G_2 are groups, and $f : G_1 \to G_2$ is a group homomorphism, then $o(G_1) = o(G_2)$.

   iii) If G is an abelian group, then G is cyclic.

   iv) If G is a group and $H \trianglelefteq G$, then $|G : H| = 2$.

   v) Every element of S_n has order at most n.

   vi) If R is a ring and I is an ideal of R, then $xr = rx \ \forall \ x \in I$ and $r \in R$.

   vii) If $\sigma \in S_n (n \geq 3)$ is a product of an even number of disjoint cycles, then sign $(\sigma) = 1$.

   viii) If a ring has a unit, then it has only one unit.

   ix) The characteristic of a finite field is zero.

   x) The set of discontinuous functions from $[0, 1]$ to $\mathbb{R}$ form a ring with respect to point-wise addition and multiplication.

2. a) Define a relation R on $\mathbb{Z}$, by $R = \{(n, n+3k) \mid k \in \mathbb{Z}\}$.
   Check whether R is an equivalence relation or not. If it is, find all the distinct equivalence classes. If R is not an equivalence relation, define an equivalence relation on $\mathbb{Z}$. (5)

   b) Consider the set $X = \mathbb{R} \setminus \{-1\}$. Define $*$ on X by
   $x_1 * x_2 = x_1 + x_2 + x_1 x_2 \forall \ x_1, x_2 \in X$.

      i) Check whether $(X, *)$ is a group or not.

      ii) Prove that $x * x * x * \ldots * x$ (n times) $= (1+x)^n - 1 \ \forall \ n \in \mathbb{N}$ and $x \in X$. (7)

   c) Give an example, with justification, of a commutative subgroup of a non-commutative group. (3)

3. a) Check whether or not $A = \{z \in \mathbb{C}^* \big| \, |z| \in \mathbb{Q}\}$ is a subgroup of

      i) $(\mathbb{C}^*, .)$, ii) $(\mathbb{C}, +)$ (3)

   b) Let $(G, .)$ be a finite abelian group and $m \in \mathbb{N}$. Prove that
   $S = \{g \in G \mid (o(g), m) = 1\} \leq G$. (3)

c) Let G be a group of order $n \geq 2$, with only two subgroups $-\{e\}$ and itself. Find a minimal generating set for G. Also, find out whether n is a prime or a composite number, or can be either. (4)

4. a) Consider the map $f_{ab} : \mathbb{R} \to \mathbb{R} : f_{ab}(x) = ax + b$. Let $B = \{f_{ab} \mid a, b \in \mathbb{R}, a \neq 0\}$. Then B is a group with respect to the composition of functions. Check whether or not $A = \{f_{ab} \mid a \in \mathbb{Q}^+, b \in \mathbb{R}\}$ is a normal subgroup of B. (4)

b) Explicitly give the elements and structure of the group S_n / A_n, $n \geq 5$. (3)

c) Let G be a group of order 56. What are all its Sylow p-subgroups? Show that G is not simple, i.e., G must have a proper normal non-trivial subgroup. (3)

5. a) Find a group G, and a homomorphism ϕ of G, so that $\phi(G) \simeq S_3$ and $\text{Ker } \phi \simeq A_4$. Is G abelian? Give reasons for your answer. (6)

b) Let G be a group such that $\text{Aut}\, G$ is cyclic. Prove that G is abelian. (4)

6. a) Check whether $I = \left\{ \begin{bmatrix} m & 0 \\ n & 0 \end{bmatrix} \middle| m, n \in \mathbb{Z} \right\}$ is a subring of the ring $\mathbb{M}_2(\mathbb{Z})$ or not. If it is, check whether or not it is an ideal of the ring also. If I is not a subring of the ring, then provide a subring of the ring. (3)

b) Prove that $\dfrac{\mathbb{R}[x]}{\langle x^2 + 1 \rangle} \simeq \mathbb{C}$ as rings. (5)

c) Find all the units of $\mathbb{Z}_{12}$. (2)

7. a) Let R be a commutative ring with unity and $r \in R$. Prove that $\dfrac{R[x]}{\langle x - r \rangle} \simeq R$ using the Fundamental Theorem of Homomorphism.
Hence show that $\dfrac{R[x, y]}{\langle y - r \rangle} \simeq R[x]$. (6)

b) Let $D = \{f(x, y) + g(x, y)\, i \mid f, g \in \mathbb{Z}[x, y]\} \subseteq \mathbb{C}[x, y]$. Check whether D is a UFD or not. (4)

8. a) Let $R = \mathbb{Z}[\sqrt{2}]$ and $M = \{a + b\sqrt{2} \in R \mid 5 \mid a \text{ and } 5 \mid b\}$.

i) Show that M is an ideal of R.

ii) Show that if $5 \nmid a$ or $5 \nmid b$, then $5 \nmid (a^2 + b^2)$, for $a, b \in \mathbb{Z}$.

iii) Hence show that if N is an ideal of R properly containing M, then $N = R$.

iv) Show that R/M is a field, and give two distinct non-zero elements of this field. (12)

b) Show that there are infinitely many values of α for which $x^7 + 15x^2 - 30x + \alpha$ is irreducible in $\mathbb{Q}[x]$. (3)