RMT-102

Ph.D. PROGRAMME IN MATHEMATICS (PHDMT)

Term-End Examination

December, 2022

RMT-102 : ANALYSIS

Time : 3 hours *Maximum Marks : 100*

Note : *Marks are indicated against each question or part thereof. Question No.* ***1*** *is* ***compulsory****. Attempt as many questions as you can from Questions No.* ***2*** *to* ***8****. The total marks awarded will be 100.*

1. Which of the following statements are *true* or *false* ? Justify you answer by giving a short proof of the statement which you think is *true* or by illustrating with counter example for the statements which are *false*. *5×2=10*

(i) A countable union of closed sets in a metric space is closed.

(ii) The function f defined by

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is measurable.

(iii) If a function is differentiable at a point, then it is analytic.

(iv) If X is a normed linear space, $x, y \in X$ and $\|x\| = 1 = \|y\|$, then $\|x + y\| \leq 2$.

(v) The transformation $T(z) = \dfrac{az + b}{cz + d}$ is a Mobius transformation only if $ad - bc \neq 0$.

2. (a) Consider $X = \mathbf{R}$ with Lebesgue measure m.

Let $1 \leq p \leq \infty$. Then

(i) Define a norm on $L^p(\mathbf{R})$ which makes it a normed linear space. Are these spaces inner product spaces ? Justify your answer.

(ii) $1 \leq p < r < \infty$ and $E \subset \mathbf{R}$ is a measurable set such that $m(E) < \infty$. Then show that $L^r(E) \subset L^p(\mathbf{R})$. *7*

(b) Let X and Y be metric spaces. Let $f : X \to Y$ be continuous. Show that $f - (V)$ is open in X for every open set V in Y. *3*

(c) State dominated convergence theorem. Use the theorem to find $\lim_{n \to \infty} \int f_n(x)\, dx$ when $f_n(x) = \dfrac{\sqrt{x}}{1 + n\, x^3}$. *5*

3. (a) (i) Define the out measure m* of a set $A \subseteq \mathbf{R}$.

(ii) Find the outer measure of the following sets :

1. $A = [3, 5] \cup \{x : x$ is a solution of the equation $x^2 + 1 = 0\}$

2. $B = \{r : r$ is a rational number in $[0, 1] \cup \mathbf{R}\backslash\mathbf{Q}$

(iii) If $E_1, E_2 \subseteq \mathbf{R}$ such that $m^*(E_1) < \infty$ and $m^*(E_2) < \infty$, then show that

$$m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2).$$

1+3+3=7

(b) Give examples of Banach algebras of which one is commutation and the other is not commutation. Justify you choice of examples. *4*

(c) Give an example of a compact set in **R** with

(i) Euclidean metric

(ii) Discrete metric

Justify your choice of examples. *4*

3. (a) If $f \in H(\Omega)$ where Ω is a domain in **C** and $Z_0 \in \Omega$ such that $f'(Z_0) \neq 0$, then show that f is conformal at Z_0. *7*

(b) Define the terms : interior, closure and boundary of any subset of a metric space. Find the interior and closure of a subset A of $\mathbf{R}^2$ where *3*

$$A = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 9\}$$

(c) When is a set $E \subset \mathbf{R}$ Lebesgue measurable ? If E_1 and E_2 are measurable sets and $E_1 \cap E_2 = \phi$, then show that $E_1 \cup E_2$ is measurable. *5*

4. (a) Define a connected set in a metric space. Check whether the following sets are connected :

(i) $A = \{(x, y) : x^2 + y^2 = 4\}$

(ii) $B = \{(x, y) : x = 0, y \geq 1\}$

Let E be a connected set in a metric space. If {A, B} is a disconnection of X, show that either $E \subseteq A$ or $E \subseteq B$. *7*

(b) State closed graph theorem. Show that the theorem may not hold if the normed linear spaces involved are not Banach spaces. *4*

(c) Prove that every Mobius transformation $T : \mathbf{C}_\infty \to \mathbf{C}_\infty$ has at most two fixed points in C_∞. *4*

5. (a) Check the measurability and integrability of the following functions defined on **R**. Justify your answer. *7*

(i) $f(x) = 2, \quad x = 1, 2, 3, 4$
$\quad\;\; = -1, x = -1, -2, -3$
$\quad\;\; = 0, \quad \text{elsewhere}$

(ii) $f(x) = x + e^x$

(iii) $f(x) = \frac{5}{2}, x \in [0, 6]$
$\quad\;\; = 0, \quad \text{elsewhere}$

(b) Suppose X is a compact metric space and Y is any metric space and $f : X \to Y$ is continuous. Prove that f(x) is compact. *4*

(c) Let X = C[0, 1], the space of all continuous functions on [0, 1]. For $f \in X$, let

$$\|f\| = \|f\|_\infty + f(1).$$

Check whether $\|\cdot\|$ defines a norm on X. *4*

6. (a) (i) State Hahn-Banach Extension theorem.

(ii) Consider the linear space $X = \mathbf{R}^2$ with $\|\cdot\|$, given by

$$\|\mathbf{x}\|_1 = |x_1| + |x_2|, \mathbf{x} = (x_1, x_2)$$

Let G denote the subspace of $\mathbf{R}^2$ given by

$$G = \{(x_1, 0) : x_1 \in \mathbf{R}\}$$

and f be the linear functional defined on G by

$$f(x_1, 0) = \alpha x, \alpha > 0.$$

Show that $\tilde{f} : X \to \mathbf{R}$ defined by

$$\tilde{f}(x) = \alpha x_1 + \frac{\alpha}{2} x_2, x = (x_1, x_2) \in X$$

is a Hahn-Banach extension of f to X. Find another extension of f to X. What is its implication on the Hahn-Banach Theorem ?

(iii) Using Hahn-Banach theorem, prove the following result :

"Let X be a normed linear space over k and $a \in X$ be such that $a \neq 0$. Then there exists a bounded linear functional on X such that

$$f(a) = \|a\|, \ \|f\| = 1$$

$$\|a\| = \text{Sup}\,\{f(a) : f \in X'\}, \ \|f\| \leq 1. \qquad 8$$

(b) Which of the following sets are closed in $\mathbf{R}^3$ with respect to the standard metric ? Justify. 3

(i) $A = \{(x, y) \in \mathbf{R}^2 : xy = 0\}$

(ii) $B = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$

(c) Consider a linear operator $T : L^1[0, 1] \to \mathbf{R}$ defined by $T(f) = \int_0^1 t\, f(t)\, dt, \ f \in L^1[0, 1]$.

Show that T is a bounded linear operation on L^1 [0, 1]. 4

7. (a) If φ and ζ are two simple functions, then show that

(i) $\int_E \varphi \, dm \leq \int_E \zeta \, dm$

(ii) $\int_{A \cup B} \varphi \, dm = \int_A \varphi \, dm + \int_B \varphi \, dm$

where A and B are disjoint measurable sets. 6

(b) Let $X = C[a, b]$. Define a function d on $X \times X \to \mathbf{R}$ by

$$d(f, g) = \int_a^b | f(t) - g(t) | dt$$

where $f, g \in X$. Show that d defines a metric on X. Does d define a metric on R[a, b], the set of Riemann integrable functions on [a, b] ? Justify your answer. *7*

(c) Show that $\{u_n\}$, where $u_n = (0, 0, \ldots 1, 0, 0 \ldots)$, 1 occurs at the n^{th} place, is an orthonormal set in l^2. *2*

8. (a) (i) Show that a metric space is complete if and only if every Cauchy sequence in it has a convergent subsequence.

(ii) Check whether a discrete metric space is complete. *8*

(b) If u and v are harmonic conjugate to each other in some domain, then show that u and v must be constant there. *7*